

# The Classical Two-Component Spinor Formalisms for General Relativity

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## Abstract

The classical two-component spinor formalisms for general relativity as built up by Infeld and van der Waerden afford an elegant approach to spacetime geometry. Deeply involved in the inner structure of these formalisms is the beautiful theory of spin densities of Schouten. In this review Schouten's theory is presented in detail. It is pointed out that spin affinities can most naively be introduced by carrying out parallel displacements of null world vectors. A complete algebraic description of spin curvatures is accomplished on the basis of the construction of a set of torsionless covariant commutators. It turns out that the implementation of such commutators under certain circumstances gives rise to a system of wave equations for gravitons and photons which possess a gauge-invariance property associated with appropriate spinor-index configurations. The situation regarding the derivation of the natural couplings between Dirac fields and spin curvatures is entertained.

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# 1 Introduction

The construction of the two-component spinor approach to specially relativistic physics was carried out a long time ago by van der Waerden [1] on the basis of the implementation of the faithful representation of the Lorentz group that is borne by the original Dirac's theory of the electron. From a more explicit mathematical point of view, this construction was based entirely upon the existence of a two-to-one homomorphism between the linear group  $SL(2, C)$  of unimodular complex  $(2 \times 2)$ -matrices and the orthochronous proper component of the Lorentz group. The space of elementary spinors accordingly emerged as a two-complex-dimensional representation space that carries an invariant antisymmetric metric spinor. A formal correspondence between Minkowskian and spin tensors was likewise accomplished from the existence of a set of Hermitian connecting objects which are subject to combined world-spin transformation laws and prescribed anticommutation relations.

Subsequently, Infeld [2] proposed a slightly extended version of the van der Waerden approach, which involves replacing the independent entry of the representative matrix for the special metric spinor with a nowhere-vanishing differentiable real-valued function defined on a generally relativistic spacetime. A relationship between this function and the functional determinant of a space-time metric tensor, together with a system of equivalent expressions for the corresponding Ricci scalar and cosmological constant, were derived from the utilization of simple spinor computational devices. These techniques took up the combination of the coordinate-derivative operator with some constant connecting objects, and thereby made it feasible to write down for the first time the two-component version of Dirac's theory in a curved background.

Soon after the presentation of Infeld's work, a geometric generalization of the special approach was exhibited by Infeld and van der Waerden [3], with two different formalisms having arisen from this generalization. Such formalisms were designated as the  $\gamma\epsilon$ -formalisms. In accordance with either of these frameworks, two pairs of conjugate spin spaces can be systematically set up at any non-singular point of a curved spacetime, but the role played by  $SL(2, C)$  had unavoidably to be taken over by a group of gauge transformations whose determinants show up as complex numbers that depend crucially upon a real parameter. It had been pointed out earlier by Weyl [4] in connection with the formulation of a generalized principle of gauge invariance that such transformations could be naturally implemented within the context of general relativity.

The  $\gamma$ -formalism version of the basic geometric objects defined formerly by van der Waerden is prescribed in such a way that a complex-valued function of spacetime coordinates is utilized in place of the real-valued metric function borne by the Infeld extension. All metric spinors for the  $\gamma$ -formalism carry an invariant character as regards the action of any manifold mapping group, and additionally behave themselves as spin tensors under the action of the Weyl gauge group. Any connecting object for the  $\gamma$ -formalism thus bears a combination of

a spin-tensor character with either a covariant or a contravariant world-vector character. The metric spinors and connecting objects for the  $\varepsilon$ -formalism are considered as entities that carry the same world characters as the ones for the  $\gamma$ -formalism. Nevertheless, a spin-density character is ascribed to each of them, whence geometric quantities generally enter into the  $\varepsilon$ -framework as spin densities. Incidentally, the theory of spin densities had already been developed by Schouten [5, 6] at the time of the advent of the generalized formalisms.

In the  $\gamma\varepsilon$ -formalisms, the specification of spin-affinity patterns rests upon the geometric properties of the usual world-affine connexions and the implementation of a requirement which ultimately amounts to taking any Hermitian connecting objects as covariantly constant entities. The procedures for building up any suitable spin connexion produce a pair of conjugate contracted spin-affine structures that carry two world-covariant quantities having different spin characters. One of these quantities appears as a world vector that eventually undergoes a local gauge transformation. It is identified with an affine electromagnetic potential that satisfies the Weyl principle, and essentially provides the imaginary parts of the contracted structures. Its physical significance depends only upon the selection of covariant derivatives for the individual  $\gamma$ -metric spinors. The other quantity emerges as the common real part of the contracted structures. In the  $\gamma$ -formalism, it is expressed up to a conventional sign as the partial derivative of the logarithm of the absolute value of an adequate metric function. Moreover, there are some particular cases in which it can be written out as an inhomogeneous spacetime contribution that allows a formal recovery of the covariance of the contracted structures for the  $\gamma$ -formalism [7]. The treatment of such cases brings forth world-spin affine connexions that are involved in the geometric structure of a well-known class of conformally flat spacetimes [8, 9]. In both formalisms, the  $\varepsilon$ -metric spinors are chosen at the outset as covariantly constant objects. In fact, this choice comes into play without affecting at all the physical characterization of any affine electromagnetic potentials. However, no metric relationship that involves the real part of a contracted spin-affine structure for the  $\varepsilon$ -formalism comes about. It appears that the rules for computing covariant derivatives of spin densities in either formalism are fixed in terms of spin-affine configurations which arise out of invoking the covariant constancy of the  $\varepsilon$ -metric spinors. Any such spin-affine computational devices are constituted by complex world-covariant prescriptions that emerge in this way.

The construction of  $\gamma\varepsilon$ -curvature structures is modelled upon the traditional procedure that considers taking commutators between covariant-derivative operators. As formulated by Infeld and Van der Waerden [3], the covariant-constancy property of the Hermitian connecting objects for both formalisms gives rise to curvature splittings which carry only the sum of purely gravitational and electromagnetic contributions. The presence of electromagnetic curvatures was bound up with the imposition of a single gauge-covariant condition upon the metric spinors for the  $\gamma$ -formalism, which is just the same as that associated with the physical significance of affine electromagnetic potentials. Unfortunately, the computational tools that were put into practice thereabout could not cope with the spinor splittings of the bivector configurations borne by the commutators utilized. Consequently, no complete description of spin curvatures was accomplished at that time. In the presence of geometric electromagnetic fields, the affine computational devices for the  $\varepsilon$ -formalism are obtained from the ones for

the  $\gamma$ -formalism by allowing for a limiting case that involves an independent  $\gamma$ -metric component. A correspondence principle associated with this limiting process can be established by looking into two systems of eigenvalue equations for the  $\gamma$ -metric spinors. Some of these equations afford the procedures that had been utilized by Infeld and van der Waerden [3] for controlling the presence or absence of electromagnetic curvatures heuristically. The imaginary part of any former device, which actually carries an electromagnetic potential for the  $\gamma$ -formalism, thus remains the same when the limiting process is carried through in some gauge frame whilst the respective former real part, which does indeed bear a spacetime-metric character, gets replaced with a physically meaningless quantity. Putting the limit into effect in the absence of geometric electromagnetic fields, yields contracted spin-affine pieces that vanish in a gauge frame. Under these latter circumstances, any affine potentials for the  $\gamma$ -formalism are expressed as useless gradients, and the  $\varepsilon$ -formalism turns out to bear a weaker meaning. It had become evident from the original work of Infeld and van der Waerden that the derivation of a set of generalized gauge transformation laws could bring about a metric principle which describes invariantly the geometric structure of the  $\gamma$ -formalism as regards the presence or absence of electromagnetic curvatures.

The Infeld-van der Waerden formalisms have been employed over the years by many authors, in several different contexts, for carrying out two-component spinor reformulations of some of the standard physical theories in both flat and curved spacetimes as well as the construction of alternative spinor patterns for classical geometric structures and a notable spinor transcription of the famous Petrov classification schemes for world-curvature tensors [8-15]. Notwithstanding the fact that the description of curvature spinors is implicitly carried by the formalisms, the spin curvatures that occur in the transcription and geometric construction we have referred to were obtained in an artificial way by just performing straightforward spinor translations of Riemann and Weyl tensors. These translational procedures have particularly led to a spinor version of Einstein's equations along with an explicit definition of wave functions for gravitons [8, 9, 15]. It has been claimed by some authors (see, e.g., Ref. [9]) that the relevance of the  $\varepsilon$ -formalism as far as curvature classifications are concerned relies upon the occurrence of a technical simplification over the Petrov schemes [16]. Based upon the local existence of null tetrads, two spin-coefficient techniques for solving Einstein's equations have also been proposed [17-20]. Somewhat surprisingly, the utmost importance of Schouten's theory of spin densities and the gauge structure inherently borne by the formalisms were both ruled out by most of the works we have cited above. According to Penrose [15, 21], the only reason for the exclusion of electromagnetic curvatures lies behind the existence of a conflicting relationship between certain spin-charge values and the Weyl group. The conflict happens when the propagation of external uncharged spinning fields is allowed for, but in contradistinction to this, the differential prescriptions supplied by the  $\gamma$ -formalism yield a description of the couplings between Dirac fields and electromagnetic curvatures without making it necessary to use a minimal coupling covariant-derivative operator [22]. Remarkably enough, any electromagnetic curvature contributions arise from affine potentials that have to be formally introduced in order to balance the overall numbers of world-spin affine components [3]. The removal of electromagnetism from the geometry therefore destroys the effectiveness of the strongly required world-spin

affine relationships. Since the formalisms impose consistently a stringent condition on the physical characterization of curvatures, the gauge behaviour of uncharged fields should be modified so as to allow their propagation to enter the descriptive frameworks.

Only recently has a fairly complete description of spin curvatures arisen [23], which brought forward what seems to be the most striking physical feature of the  $\gamma\varepsilon$ -formalisms, namely, the possible occurrence of wave functions for gravitons and photons in the curvature structures of generally relativistic spacetimes. This insight has sprung partially from the achievement of some of the most significant developments of the spinor calculational methods, which are intimately related to the construction of sets of algebraic expansions and formal valence-reduction devices [8, 9]. Such techniques had initially afforded a cosmological interpretation of the spinors that occur in the translation of Riemann tensors [8]. One of their important properties is that they are applicable equally well to specially and generally relativistic situations because of their symbolic character. Loosely speaking, geometric photons are described by wave functions that amount to contracted spin-curvature pieces borne by spinor decompositions of Maxwell bivectors. Wave functions for gravitons are prescribed as totally symmetric curvature pieces that occur in spinor representations of Weyl tensors. The definition of these gravitational contributions thus coincides with the one mentioned before, but their full algebraic characterization has always to be made up by world configurations [14]. In a spacetime that admits non-vanishing electromagnetic and gravitational wave functions, background photons interact with underlying gravitons, with the occurrent couplings turning out to be in both formalisms exclusively borne by the equations which control the electromagnetic propagation [24]. Indeed, it is the spinor decomposition of a set of covariant commutators for both formalisms that makes up the description of curvature splittings. The pertinent computations take up the utilization of differential prescriptions which specify the action of the commutators on arbitrary spin tensors and densities. In the presence of electromagnetic curvatures, the implementation of these commutators leads to a system of wave equations for gravitons and geometric photons which possess in either formalism a gauge-invariance property associated with appropriate spinor-index configurations.

The present work is just aimed at supplying a self-consistent description of the inner structure of the  $\gamma\varepsilon$ -formalisms. Hence, attention will be concentrated upon the description of the key structures associated to the fundamental role played by spin densities in both the formalisms. The gauge behaviour of any admissible spin connexions is described conjunctively with a covariant description of the limiting process. We will exhibit the entire description of curvatures upon paving the way for deriving the system of wave equations for gravitons and geometric photons. We will likewise derive the patterns that describe the propagation of Dirac fields, in conformity with one of the original motivations of Infeld and van der Waerden for constructing the formalisms. We have divided the whole work into four Sections. The detailed descriptions and outlines of Sections 2 and 3 will be given in due course. In Section 4, we make some remarks on the formalisms. We have decided from the beginning to adopt the following conventions. Greek and Latin letters are broadly used as kernel letters for world and spin quantities in a curved spacetime  $\mathfrak{M}$ . Kernel letters for world densities will especially appear as Gothic letters. A horizontal bar lying over a kernel letter will denote the ordinary operation of complex conjugation. Unprimed

and primed kernel letters will be used to refer to outcomes of gauge transformations. Components of world and spin quantities are, respectively, labelled by lower-case and upper-case Latin letters. The unprimed-primed-index notation of Bach [25] and Schouten [6, 26] will be applied to the case of conjugate spinor components. World indices all range over the four values 0, 1, 2, 3 whereas spinor indices take either the values 0, 1 or 0', 1'. We will adhere to Bach's convention [25], according to which the effect on any index structure of the actions of the symmetry and antisymmetry operators is indicated by surrounding the relevant indices with round and square brackets, respectively. Vertical bars surrounding an index block will mean that the indices singled out are not to partake of a symmetry operation. Any world quantity having  $p$  upper and  $q$  lower indices will sometimes be referred to as a quantity of valence  $\{p, q\}$ . Similarly, a spinor carrying  $a$  upper and  $b$  lower unprimed indices together with  $c$  upper and  $d$  lower primed indices will be termed as a spinor of valence  $\{a, b; c, d\}$ . The symbol "c.c." will denote an overall complex-conjugate piece. For convenience, the partial-derivative operator  $\partial/\partial x^a$  for some spacetime coordinates  $x^0, x^1, x^2$  and  $x^3$  on  $\mathfrak{M}$  will be written as  $\partial_a$ . Without any risk of confusion, we will utilize a torsionless operator  $\nabla_a$  upon dealing with covariant derivatives in each formalism. Throughout the work, it will be assumed that the spacetime metric signature is  $(+ - - -)$ . Use will be made of the natural system of units wherein  $c = \hbar = 1$ . We will continue using the words *object* and *quantity* without calling upon any conceptual specifications like those fixed up by Schouten [27]. Further conventions will be explained occasionally.

## 2 Spin-Affine Geometry

A natural procedure for bringing covariant spinor differentials in  $\mathfrak{M}$  consists in carrying out parallel displacements from one spin space to another, which absorb the same geometric definitions as the ones for the world situation [6, 26, 27]. It appears that world-spin affine correlations and covariant-derivative prescriptions can most easily be attained by combining the covariant constancy of the Hermitian connecting objects of the  $\gamma$ -formalism and the covariant Leibniz expansion of an adequate spin-tensor outer product associated to a null world vector [23]. In fact, Infeld and Van der Waerden [3] had realized that contracted spin affinities carrying nowhere-vanishing real and imaginary parts should be taken up by the  $\gamma\varepsilon$ -frameworks because of the strong necessity of balancing the numbers of independent world-spin affine components. The expression for a generic spin affinity of either formalism is consequently obtained by first performing an appropriate index splitting of the Christoffel connexion  $\Gamma_{abc}$  for a covariant metric tensor  $g_{ab}$  on  $\mathfrak{M}$ , and then calling for world covariant-derivative patterns. Any allowable spin-affine connexion is thus made out of the spinor versions of both  $\Gamma_{a[bc]}$  and  $\Gamma_{a(bc)}$ . In either formalism, the former  $\Gamma$ -contribution supplies the symmetric part of a two-piece spinor splitting which has to be added to a non-Hermitian partial derivative. The trace  $\Gamma_a$  of the latter  $\Gamma$ -contribution noticeably makes up a general scalar-density prescription for a basic  $\gamma$ -metric function. A recovery of the real part of a contracted spin affinity for the  $\gamma$ -formalism can be accomplished from such configurations, but the feasibility of such a recovery ceases happening when the metric limiting situation that yields the affine computational devices for the  $\varepsilon$ -formalism is implemented.

The information carried by the metric spinors of the  $\gamma$ -formalism is oftenly extracted from their partial derivatives and brought out by a set of world-covariant vectors [3, 11]. One then becomes able to derive a covariant differential relationship between the metric quantities of the  $\gamma$ -formalism and the parts of the respective contracted spin-affine structures. The absolute value and polar argument of the complex-valued function that defines a  $\gamma$ -metric component accordingly appear as world scalars. It has been proven by Cardoso [23, 28] that the absolute value must be effectively expressed as the product of two scalar densities. Whereas the information on one of these densities is totally contained in a suitably contracted partial derivative of an Hermitian connecting object for the  $\gamma$ -formalism, the information on the other is carried by the determinant  $\mathfrak{g}$  of  $g_{ab}$ , with the former density having to be thought of as bearing a double world-spin character.

Before completing the geometric specifications of the metric spinors and connecting objects for the  $\varepsilon$ -formalism, one has to recall that any non-vanishing totally antisymmetric spin quantity is proportional in either formalism to one of the corresponding metric spinors. Such specifications come all from the establishment of the gauge transformation laws for the metric spinors of both formalisms. The usual definition of spin densities is shaped upon the one which is adopted in the world framework. It turns out that all metric and spin-affine prescriptions have to be combined together with the world invariance of the metric spinors. The full geometric characterization of the systems of eigenvalue equations mentioned in Section 1, rests upon the combination of the covariant constancy of  $g_{ab}$  with the standard relationships between the metric and connecting objects for the  $\gamma$ -formalism. We will emphasize that the eigenvalues carried by these equations may supply a technique for controlling the gauge behaviours of the quantities involved in the limiting process. It will likewise be seen that the procedures concerning the specification of the gauge behaviours of spin-affine connexions, afford a differential device which enables one to mix up and keep track of gauge frames when computing covariant derivatives in the  $\gamma$ -formalism.

It will be necessary to bring in Subsection 2.1 the definitions of the metric spinors and connecting objects for both formalisms. In Subsection 2.2, the gauge behaviours of the basic objects for the  $\varepsilon$ -formalism are specified in conjunction with the definition of spin tensors and densities. We shall have to include the definition of densities that bear a combined world-spin character because of the occurrence of such a quantity in the expression for a typical  $\gamma$ -metric component. The spin-affine structures are shown in Subsection 2.3 along with the relevant covariant-derivative patterns and computational devices. All eigenvalue equations and metric expressions are deduced in Subsection 2.4. The transformation laws for spin-affine connexions and the gauge description of the limiting process are considered together in Subsection 2.5. Gothic letters will henceforth also be used to denote weights of spin densities. It will be understood from now on that world-gauge characters are intrinsic geometric attributes which must not as such depend upon the implementation of any differential configurations. In Subsections 2.3 and 2.4, we shall make particular use of the relations [29-31]

$$\Gamma_a \doteq \Gamma_{ab}{}^b = \partial_a \log(-\mathfrak{g})^{1/2}.$$

## 2.1 Metric Spinors and Connecting Objects

The fundamental metric spinor of the  $\gamma$ -formalism is taken as a spin tensor of valence  $\{0, 2; 0, 0\}$ , which bears antisymmetry and invariance under world-coordinate transformations. In other words,

$$(\gamma_{AB}) = \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix}, \quad \gamma = |\gamma| \exp(i\Phi), \quad (2.1)$$

where the entries of the pair  $(|\gamma|, \Phi)$  are smooth real-valued functions of  $x^a$ . The inverse of  $(\gamma_{AB})$  appears as a world-invariant spin tensor of valence  $\{2, 0; 0, 0\}$ , which is set as

$$(\gamma^{AB}) = \begin{pmatrix} 0 & \gamma^{-1} \\ -\gamma^{-1} & 0 \end{pmatrix}. \quad (2.2)$$

One has the component relationships

$$\gamma_{AB} = \gamma \varepsilon_{AB}, \quad \gamma^{AB} = \gamma^{-1} \varepsilon^{AB}, \quad (2.3)$$

with

$$(\varepsilon_{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (\varepsilon^{AB}) \quad (2.4)$$

being the only unprimed metric spinors for the  $\varepsilon$ -formalism, which are likewise taken to bear world invariance. Thus, the independent component  $\gamma$  of  $\gamma_{AB}$  presumably carries a world-invariant character.<sup>1</sup> Equations (2.1)-(2.4) imply that

$$M^{CB} M_{CA} = M_A{}^B = -M^B{}_A, \quad (2.5a)$$

where the kernel letter  $M$  stands here as elsewhere for either  $\gamma$  or  $\varepsilon$ , and

$$(M_A{}^B) \doteq (\delta_A{}^B) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (2.5b)$$

The metric spinors and their complex conjugates serve particularly for lowering and raising indices of arbitrary spinor and world-spin quantities. For some elementary spinor  $\nu^A$ , for instance, we have the upper-lower-index prescriptions

$$\nu^A = M^{AB} \nu_B, \quad \nu_A = \nu^B M_{BA}. \quad (2.6)$$

The processes of lowering and raising spinor indices in the  $\gamma$ -formalism always preserve intrinsic spin characters because of the spin-tensor character of the metric configurations (2.1) and (2.2). It will be stressed in Subsection 2.2 that the action of the  $\varepsilon$ -metric spinors does not, in general, retain the spin characters of the former objects. However, in view of the world invariance of the structures (2.1)-(2.4), the world characters of any spin objects will remain unchanged as we implement the action of the metric spinors for either formalism.

The connecting objects of the  $\gamma$ -formalism are defined as

$$2\sigma_{AA'}(a\sigma_b^{BA'}) = \gamma_A{}^B g_{ab}, \quad (2.7a)$$

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<sup>1</sup>The uniqueness of the spinors given by Eq. (2.4) will be described in Subsection 2.2 together with the gauge behaviour of  $\gamma$ .

or, alternatively, as the complex conjugate of Eq. (2.7a). Similarly, for the  $\varepsilon$ -formalism, we have

$$2\Sigma_{AA'(a)}\Sigma_b^{BA'} = \varepsilon_A{}^B g_{ab}. \quad (2.7b)$$

All entries of the set

$$\mathbf{H} = \{S_{aAA'}, S_{AA'}^a, S_a^{AA'}, S^{aAA'}\} \quad (2.8)$$

are components of Hermitian  $(2 \times 2)$ -matrices<sup>2</sup> that depend smoothly upon  $x^a$ . We should notice that the Hermiticity of any element of the set (2.8) is lost when we let its spinor indices share out both stairs. Hence, manipulating the indices of Eqs. (2.7) suitably, and symmetrizing over  $AB$ , yields the property

$$S_{aA'}^{(A} S_b^{B)A'} = S_{A'[a}^{(A} S_{b]}^{B)A'} = S_{A'[a}^A S_{b]}^{BA'}, \quad (2.9a)$$

and, consequently, we can write

$$S_{AA'[a} S_b^{AA'} = 0 \Leftrightarrow S_{AA'a} S_b^{AA'} = S_{AA'(a} S_{b)}^{AA'}. \quad (2.9b)$$

The index configurations of Eqs. (2.9) can be worked out so as to give the contracted commutator

$$[S_{aA'}^A, S_b^{BA'}] = 0, \quad (2.10a)$$

which leads to the relations

$$S_{A'}^{a(A} S_a^{B)A'} = 0 \Leftrightarrow S_{A'}^{aA} S_a^{BA'} = 2M^{AB}. \quad (2.10b)$$

In either formalism, the pertinent  $S$ -objects provide a one-to-one correspondence between world and spin objects, which is written in terms of adequate outer products.<sup>3</sup> Some metric examples are the following:

$$g_{ab} = S_a^{AA'} S_b^{BB'} M_{AB} M_{A'B'}, \quad (2.11a)$$

and

$$M_{AB} M_{A'B'} = S_{AA'}^a S_{BB'}^b g_{ab}. \quad (2.11b)$$

Thus, one of the spinor structures that represent the alternating *tensors* on  $\mathfrak{M}$  is expressed by [8, 15, 32]

$$e_{AA'BB'CC'DD'} = i(M_{AC} M_{BD} M_{A'D'} M_{B'C'} - \text{c.c.}), \quad (2.12a)$$

which agrees with the trivial identities

$$M_{[AB} M_{C]D} \equiv 0, \quad (2.12b)$$

and

$$M_{A(B} M_{C)D} = M_{B(A} M_{D)C}. \quad (2.12c)$$

The combination of Eqs. (2.3) and (2.11) evidently produces the Hermitian associations

$$\sigma_{AA'}^a = |\gamma| \Sigma_{AA'}^a, \quad \sigma^{aAA'} = |\gamma|^{-1} \Sigma^{aAA'}, \quad (2.13a)$$

<sup>2</sup>From this time onwards, the kernel letter  $S$  will denote either  $\sigma$  or  $\Sigma$ .

<sup>3</sup>This correspondence does not apply to  $x^a$ , but it naturally applies to  $\partial_a$  and  $dx^a$ .

along with the lower-world-index ones. An example of a  $\sigma\Sigma$ -association in the non-Hermitian case is provided by

$$\sigma_{aA}^{A'} = \exp(i\Phi)\Sigma_{aA}^{A'}. \quad (2.13b)$$

It was said in Section 1 that any connecting object for either formalism is thought of as a vector as regards world-coordinate transformations, whence any outer products of  $S$ -objects must bear a world-tensor character. It follows that any spinor associated to a world tensor behaves as a scalar in relation to transformations belonging to the manifold mapping group of  $\mathfrak{M}$ . Likewise, since all the connecting objects for the  $\gamma$ -formalism are considered as spin tensors as well, any couplings of  $\sigma$ -objects with purely world quantities must yield spin tensors, but this generally fails to hold for the case of the  $\varepsilon$ -formalism.

## 2.2 Spin Tensors and Densities

The generalized Weyl gauge group consists of the set of all non-singular complex  $(2 \times 2)$ -matrices  $(\Lambda_A^B)$  whose components are prescribed as [3, 4, 11]

$$\Lambda_A^B = \sqrt{\rho} \exp(i\Lambda) \delta_A^B. \quad (2.14a)$$

In Eq. (2.14a),  $\rho$  is a positive-definite differentiable real-valued function of  $x^a$  and  $\Lambda$  amounts to the gauge parameter of the group, which is taken as an arbitrary differentiable real-valued function on  $\mathfrak{M}$ . This group operates locally on the spin spaces of  $\mathfrak{M}$ , independently of the effective action of the spacetime mapping group. For the determinant of  $(\Lambda_A^B)$ , we then have the expression

$$\det(\Lambda_A^B) \doteq \Delta_\Lambda = \rho \exp(2i\Lambda), \quad (2.14b)$$

whence

$$\Lambda_A^B \Lambda_C^D = \Delta_\Lambda \delta_A^B \delta_C^D, \quad (2.14c)$$

and  $\rho \doteq |\Delta_\Lambda|$ . By definition, one of the simplest indexed spin tensors is an unprimed covariant spin vector which undergoes the transformation law

$$\xi'_A = \Lambda_A^B \xi_B. \quad (2.15a)$$

Hence, requiring the inner product  $\zeta^A \xi_A$  to bear gauge invariance, yields the basic unprimed contravariant law

$$\zeta'^A = \zeta^B \Lambda_B^{-1A}. \quad (2.15b)$$

Obviously, the transformation laws for primed spin vectors take up either the complex conjugate matrix  $(\Lambda_{A'}^{B'})$  or its inverse.<sup>4</sup>

The defining transformation laws for spin tensors of arbitrary valences are usually obtained by performing outer products between spin vectors and applying appropriately the prescriptions (2.15). Thus, the spin-tensor character of the metric and connecting objects for the  $\gamma$ -formalism is brought out by the covariant and contravariant configurations

$$\gamma'_{AB} = \Lambda_A^C \Lambda_B^D \gamma_{CD}, \quad (2.16a)$$

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<sup>4</sup>Any spin scalar is defined as a numerical quantity that is invariant under gauge transformations.

$$\gamma'^{AB} = \gamma^{CD} \Lambda_C^{-1A} \Lambda_D^{-1B}, \quad (2.16b)$$

and

$$\sigma'_{AA'}^a = \Lambda_A^B \Lambda_{A'}^{B'} \sigma_{BB'}^a, \quad (2.17a)$$

$$\sigma'^{aAA'} = \sigma^{aBB'} \Lambda_B^{-1A} \Lambda_{B'}^{-1A'}, \quad (2.17b)$$

along with their complex conjugates and lower-world-index versions. By virtue of Eq. (2.14c), the laws (2.16) and (2.17) can be rewritten as

$$\gamma'_{AB} = \Delta_\Lambda \gamma_{AB}, \quad \gamma'^{AB} = \delta_\Lambda \gamma^{AB}, \quad (2.18)$$

and

$$\sigma'_{AA'}^a = |\Delta_\Lambda| \sigma_{AA'}^a, \quad \sigma'^{aAA'} = |\Delta_\Lambda|^{-1} \sigma^{aAA'}, \quad (2.19)$$

with  $\delta_\Lambda \doteq (\Delta_\Lambda)^{-1}$ . For the non-Hermitian  $\sigma$ -objects, we have, for instance,

$$\sigma'_{aA'}^B = \Lambda_{A'}^{B'} \sigma_{aB'}^C \Lambda_C^{-1B} = \exp(-2i\Lambda) \sigma_{aA'}^B. \quad (2.20)$$

Inasmuch as the spin spaces of  $\mathfrak{M}$  are all two-dimensional, the only useful totally antisymmetric spin objects bear two indices of the same type. In the spin-tensor case, such an object  $\eta_{AB}$  may be expanded as

$$\eta_{AB} = \eta_{[AB]} = \frac{1}{2} \eta \gamma_{AB}, \quad (2.21)$$

with  $\eta = \eta_C^C$  thus being a spin scalar.<sup>5</sup> The original definitions of complex spin-scalar densities of weights +1 and -1 were naively designed [5, 6, 26] from transformation laws that look like Eqs. (2.18). Such entities thus undergo the same gauge transformation laws as the individual independent components of  $\gamma_{AB}$  and  $\gamma^{AB}$ , respectively. For a complex spin-scalar density  $\alpha$  of weight  $\mathfrak{w}$ , one has the definition<sup>6</sup>

$$\alpha' = (\Delta_\Lambda)^{\mathfrak{w}} \alpha. \quad (2.22)$$

It is clear that the operation of complex conjugation on spin-scalar densities can be described as an interchange involving the non-vanishing unprimed and primed  $\gamma$ -metric components. The complex conjugate of  $\alpha$  is called a spin-scalar density of antiweight  $\mathfrak{w}$ . Performing outer products between spin-scalar densities produces other densities whose weights and antiweights equal the sums of the corresponding attributes carried by the couplings. Therefore, a spin-scalar density  $\beta$  of weight  $\mathfrak{a}$  and antiweight  $\mathfrak{b}$  must transform as

$$\beta' = (\Delta_\Lambda)^{\mathfrak{a}} (\bar{\Delta}_\Lambda)^{\mathfrak{b}} \beta. \quad (2.23a)$$

When  $\mathfrak{a} = \mathfrak{b}$ , the density  $\beta$  is particularly said to bear an absolute weight  $2\mathfrak{a}$ , and thence behaves itself under gauge transformations as

$$\beta' = |\Delta_\Lambda|^{2\mathfrak{a}} \beta. \quad (2.23b)$$

Then, spin-scalar densities of absolute weights  $\pm 1$  are subject to the same transformation laws as the components of the connecting objects involved in Eqs. (2.19). The pattern (2.23a) may be specialized still further in case Hermiticity

<sup>5</sup>Statements similar to Eqs. (2.21) also hold for the contravariant and primed cases.

<sup>6</sup>Our  $\Delta_\Lambda$  is the inverse of that used in Ref. [27].

is required to be preserved under gauge transformations. Consequently, any real spin-scalar density must bear an absolute weight. As will become manifest later, it is of some interest to take into consideration spin-scalar densities that bear weights, antiweights as well as absolute weights. For such a composite density  $\Omega$ , we have the prescription

$$\Omega' = (\Delta_\Lambda)^a (\bar{\Delta}_\Lambda)^b | \Delta_\Lambda |^c \Omega. \quad (2.24)$$

Arbitrary spin-tensor densities were originally defined [5, 6] as outer products between spin tensors and scalar densities, in formal analogy with the world situation. Conventionally, the entries of the arrays that supply the valences of these outer-product structures are given as the sums of the corresponding entries of the valences borne by the coupled tensors, while the overall weights and antiweights are prescribed in the same way as for coupled spin-scalar densities. In particular, any Hermitian spin-tensor density must be viewed as the product of an Hermitian tensor with a real spin-scalar density. Of course, we can build up spin tensors by performing products that carry suitable spin scalar and tensor densities. Configurations that possess a mixed world-spin density character can also be constructed by performing outer products between world and spin densities.<sup>7</sup>

The easiest procedure for specifying the gauge characters of the  $\varepsilon$ -metric spinors involves the combination of Eqs. (2.3) and (2.18). In effect, we have the laws

$$\varepsilon'_{AB} = (\Delta_\Lambda)^{-1} \Lambda_A^C \Lambda_B^D \varepsilon_{CD} = \varepsilon_{AB}, \quad (2.25a)$$

and

$$\varepsilon'^{AB} = \Delta_\Lambda \varepsilon^{CD} \Lambda_C^{-1A} \Lambda_D^{-1B} = \varepsilon^{AB}, \quad (2.25b)$$

along with their complex conjugates. Hence  $(\varepsilon_{AB}, \varepsilon^{AB})$  and  $(\varepsilon_{A'B'}, \varepsilon^{A'B'})$  are invariant spin-tensor densities of weights  $(-1, +1)$  and antiweights  $(-1, +1)$ , respectively. Any metric spinor for the  $\varepsilon$ -formalism can then be naturally considered as a spin Levi-Civita symbol. It should be stressed that  $\Delta_\Lambda$  is formally expressed in both formalisms as

$$\Delta_\Lambda = \frac{1}{2} M^{AB} \Lambda_A^C \Lambda_B^D M_{CD}. \quad (2.26)$$

Whereas the metric components  $(\gamma, \gamma^{-1})$  and  $(\bar{\gamma}, \bar{\gamma}^{-1})$  thus have to be regarded as spin-scalar densities of weights  $(+1, -1)$  and antiweights  $(+1, -1)$ , the absolute values  $(|\gamma|, |\gamma|^{-1})$  must be taken as real spin-scalar densities of absolute weights  $(+1, -1)$ , respectively. In addition, the polar piece  $\exp(i\Phi)$  of  $\gamma$  must behave as a composite spin-scalar density, namely,

$$\exp(i\Phi') = \Delta_\Lambda | \Delta_\Lambda |^{-1} \exp(i\Phi). \quad (2.27)$$

Accordingly, Eqs. (2.13a) imply that  $\Sigma_{aAA'}$  and  $\Sigma_a^{AA'}$  should behave as invariant spin-tensor densities of absolute weights  $-1$  and  $+1$ , respectively. More explicitly, we have

$$\Sigma'_{aAA'} = | \Delta_\Lambda |^{-1} \Lambda_A^B \Lambda_{A'}^{B'} \Sigma_{aBB'} = \Sigma_{aAA'}, \quad (2.28a)$$

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<sup>7</sup>Particularly interesting world-spin scalar densities have the form  $(-\mathfrak{g})^N \alpha$ , with  $N$  being any real number.

and

$$\Sigma_a'^{AA'} = |\Delta_\Lambda| \Sigma_a^{BB'} \Lambda_B^{-1A} \Lambda_{B'}^{-1A'} = \Sigma_a^{AA'}. \quad (2.28b)$$

We can now see that the implementation of Eqs. (2.16) ensures the preservation of spin characters when the processes of lowering and raising spinor indices take place in the  $\gamma$ -formalism. In turn, Eqs. (2.25) and their complex conjugates show us that the change in the  $\varepsilon$ -formalism of the spinor-index configuration of an arbitrary spin object generally produces a modification of the values of the pertinent weights and antiweights. Hence, correspondences within the  $\varepsilon$ -framework between world and spin quantities do not generally involve spin tensors.

### 2.3 Spin Affinities and Covariant Derivatives

The patterns of spin displacements were originally chosen [3, 6, 11, 13, 26, 27] so as to resemble closely the form borne by the ones which occur in the purely world framework. We thus consider two neighbouring spin spaces of  $\mathfrak{M}$  which are set up at  $x^a$  and  $x^a + dx^a$ . A covariant differential of some contravariant spin vector  $\zeta^A$  is defined as the local difference between the value of  $\zeta^A$  at  $x^a + dx^a$  and the value at  $x^a$  of the spin vector that results from an affine displacement of  $\zeta^A$ . In either formalism, a typical covariant-differential configuration reads

$$D\zeta^A = d\zeta^A + \vartheta_{aB}{}^A \zeta^B dx^a, \quad (2.29)$$

with  $\vartheta_{aB}{}^A$  amounting to the unprimed-index spin-affine connexion associated to the displacement eventually carried out. For the corresponding covariant derivative, we have

$$\nabla_a \zeta^A = \partial_a \zeta^A + \vartheta_{aB}{}^A \zeta^B. \quad (2.30)$$

Either  $D$ -differential of a covariant spin vector  $\xi_A$  can be obtained from Eq. (2.29) by taking for granted the Leibniz rule and demanding that

$$D(\zeta^A \xi_A) = d(\zeta^A \xi_A), \quad (2.31)$$

whence we also have<sup>8</sup>

$$\nabla_a \xi_A = \partial_a \xi_A - \vartheta_{aA}{}^B \xi_B, \quad (2.32)$$

together with the complex conjugates of the prescriptions (2.30) and (2.32). We stress that each of the pieces which occur on the right-hand sides of Eqs. (2.30) and (2.32) must behave covariantly under the action of the mapping group of  $\mathfrak{M}$ , in contrast with the world situation.

World and spin displacements in  $\mathfrak{M}$  turn out to be induced by each other when the covariant-constancy requirement

$$DS_{AA'}^a = 0 \quad (2.33)$$

is implemented. Whenever a tensor quantity that carries both world and spin indices is differentiated covariantly in both formalisms, we will thus have to incorporate into the pertinent expansions the affine contributions associated with all the indices borne by the quantity being considered. Any such mixed

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<sup>8</sup>As for the world case, covariant derivatives of spin tensors of arbitrary valences can always be obtained by allowing for outer products between elementary spin vectors and carrying out Leibniz expansions thereof.

expansion must be regarded as a result of the implementation of combined world-spin displacements in  $\mathfrak{M}$ . The simplest procedure for establishing this geometric property of the formalisms just accounts for affine displacements of the following  $\gamma$ -formalism configuration:

$$n^a = \sigma_{AA'}^a \zeta^A \zeta^{A'}, \quad n^a n_a = 0. \quad (2.34a)$$

Hence, writing

$$Dn^a = \sigma_{AA'}^a D(\zeta^A \zeta^{A'}), \quad (2.34b)$$

and performing a Leibniz expansion, yields the correlation [23]

$$\Gamma_{bc}{}^a n^b dx^c = \sigma_{AA'}^a (\gamma_{bB}{}^A \zeta^B \zeta^{A'} + \text{c.c.}) dx^b - \zeta^A \zeta^{A'} d\sigma_{AA'}^a, \quad (2.34c)$$

with  $\gamma_{aA}{}^B$  standing for the  $\gamma$ -formalism version of  $\vartheta_{aA}{}^B$ . It becomes clear that Eq. (2.33) should be spelt out here as the vanishing derivative

$$\nabla_a \sigma_{BB'}^b = \partial_a \sigma_{BB'}^b + \Gamma_{ac}{}^b \sigma_{BB'}^c - (\gamma_{aB}{}^C \sigma_{CB'}^b + \text{c.c.}). \quad (2.35)$$

Differentiating covariantly both sides of Eq. (2.11a) then brings about the particular metric condition

$$\nabla_a g_{BB'CC'} = \nabla_a (\gamma_{BC} \gamma_{B'C'}) = 0, \quad (2.36)$$

and, consequently, also its upper-spinor-index version. It follows that any Hermitian connecting object for the  $\gamma$ -formalism bears covariant constancy, whence we have the relation

$$\text{Re}(\gamma^{BC} \nabla_a \gamma_{BC}) = 0, \quad (2.37)$$

together with the one which is obtained from Eq. (2.37) by interchanging the spinor-index stairs. Since  $\nabla_a \delta_C{}^D = 0$ , we also obtain

$$\gamma^{BD} \nabla_a \gamma_{BC} + \gamma_{BC} \nabla_a \gamma^{BD} = 0. \quad (2.38)$$

In both formalisms, Eq. (2.33) ensures a recovery of covariant-differential patterns for world tensors from those for Hermitian spin tensors. It becomes imperative in any case to regularize the number of spin-affine components so as to attain a compatible relationship with the 40 independent components of  $\Gamma_{abc}$ . The index configuration of  $\vartheta_{aA}{}^B$  supplies 32 real independent components, whence the contracted structure  $\vartheta_{aB}{}^B$  has to carry explicitly non-vanishing real and imaginary parts. In the  $\gamma$ -formalism, the real part automatically arises when we invoke Eqs. (2.3) together with their complex conjugates for working out the condition (2.36). We have, in effect,

$$\nabla_a (\gamma_{BC} \gamma_{B'C'}) = (\partial_a \log |\gamma|^2 - 2 \text{Re} \gamma_{aD}{}^D) \gamma_{BC} \gamma_{B'C'}, \quad (2.39)$$

which produces the constraint<sup>9</sup>

$$\text{Re} \gamma_{aB}{}^B = \partial_a \log |\gamma|. \quad (2.40)$$

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<sup>9</sup>It should be noticed that the right-hand side of Eq. (2.40) bears world covariance as  $\gamma$  is a world scalar by definition. However, we can not rewrite it by replacing  $\partial_a$  with  $\nabla_a$  because of the spin-density character of  $\gamma$ .

The original regularization procedure for the  $\gamma$ -formalism [3] was carried through by implementing by hand a make-up constraint for  $\gamma_{aB}{}^B$  that involves a prescription of the type

$$\text{Im } \gamma_{aB}{}^B = (-2)\Phi_a, \quad (2.41)$$

with  $\Phi_a$  being a world vector. What should be emphatically observed in respect of this situation is that covariant differentials in the  $\gamma$ -formalism of any Hermitian  $\sigma$ -objects, and thence also Eq. (2.39) itself, remain all unaffected<sup>10</sup> when purely imaginary world-covariant quantities like  $i\tau_a\delta_B{}^C$  are added to  $\gamma_{aB}{}^C$ . Consequently, combining Eqs. (2.40) and (2.41) yields the structure

$$\gamma_{aB}{}^B = -(\theta_a + 2i\Phi_a), \quad (2.42)$$

with the definition

$$\theta_a \doteq \partial_a \log(|\gamma|^{-1}). \quad (2.43)$$

When dealing with covariant differentiations in  $\mathfrak{M}$ , we thus have to call for the affine relationship

$$\Gamma_{AA'BB'CC'} + \sigma_{hCC'}\partial_{AA'}\sigma_{BB'}^h = \gamma_{AA'BC}\gamma_{B'C'} + \text{c.c.}, \quad (2.44a)$$

along with the splittings

$$\sigma_{AA'}^a\sigma_{BB'}^b\sigma_{CC'}^c\Gamma_{a(bc)} = \Gamma_{A(BC)A'(B'C')} + \frac{1}{4}\Gamma_{AA'}\gamma_{BC}\gamma_{B'C'}, \quad (2.44b)$$

and

$$\sigma_{AA'}^a\sigma_{BB'}^b\sigma_{CC'}^c\Gamma_{a[bc]} = \Theta_{AA'BC}\gamma_{B'C'} + \text{c.c.}, \quad (2.44c)$$

where

$$\Gamma_{A(BC)A'(B'C')} = \sigma_{AA'}^a\sigma_{BB'}^b\sigma_{CC'}^c\hat{s}\Gamma_{a(bc)}, \quad (2.44d)$$

$$\Gamma_{AA'} = \sigma_{AA'}^a\Gamma_a, \quad (2.44e)$$

and

$$2\Theta_{AA'BC} = \sigma_{AA'}^a\sigma_{(B}^{bD'}\partial_{C)D'}g_{ab} = \Gamma_{A(BC)A'M'}{}^{M'} = 2\Theta_{AA'(BC)}, \quad (2.44f)$$

with the purely world kernel of Eq. (2.44d) being given by the trace-free relation<sup>11</sup>

$$\hat{s}\Gamma_{a(bc)} \doteq \Gamma_{a(bc)} - \frac{1}{4}\Gamma_ag_{bc}. \quad (2.44g)$$

One can now manipulate the index configuration of Eq. (2.44a) to produce the formulae

$$\Gamma_{A(BC)A'(B'C')} = -\sigma_{AA'}^a\sigma_{h(B(B'\partial_{|a|}\sigma_{C')C}^h)}, \quad (2.45a)$$

$$\gamma_{a(BC)} = \Theta_{aBC} + \frac{1}{2}\sigma_{h(B}^{B'}\partial_{|a|}\sigma_{C)B'}^h, \quad (2.45b)$$

and

$$4\text{Re } \gamma_{aB}{}^B = \Gamma_a + \sigma_h^{BB'}\partial_a\sigma_{BB'}^h, \quad (2.45c)$$

<sup>10</sup>This applies to the  $\varepsilon$ -framework as well. The regularization procedure for the  $\varepsilon$ -formalism will be entertained later in this Section.

<sup>11</sup>The operator  $\hat{s}$  picks out linearly the trace-free parts of any two-world-index configurations (see Ref. [8]).

together with the complex conjugate of Eq. (2.45b).

For checking upon the legitimacy of the splitting (2.44b), it is convenient to make use of the definition (2.43) to write the statement

$$\sigma_{BB'}^b \sigma_{CC'}^c \partial_{AA'} g_{bc} = -[2\theta_{AA'} \gamma_{BC} \gamma_{B'C'} + g_{bc} \partial_{AA'} (\sigma_{BB'}^b \sigma_{CC'}^c)]. \quad (2.46a)$$

Equations (2.9) imply that the products of  $g_{bc}$  with the partial derivatives of the crossed pieces

$$(\sigma_{[B[B']\sigma_{C']C]}^b, \sigma_{[B(B')\sigma_{C']C]}^b) \quad (2.46b)$$

both vanish, whereas the product that carries the partial derivative of the totally symmetric piece is given by

$$g_{bc} \partial_{AA'} (\sigma_{(B(B')\sigma_{C']C]}^b) = (-2) \Gamma_{A(BC)A'(B'C')}. \quad (2.46c)$$

For the contribution that involves the totally antisymmetric piece, we have the expression

$$g_{bc} \partial_{AA'} (\sigma_{[B[B']\sigma_{C']C]}^b) = \frac{1}{4} |\gamma|^{-2} \gamma_{BC} \gamma_{B'C'} g_{bc} \partial_{AA'} (|\gamma|^2 g^{bc}), \quad (2.46d)$$

whence, fitting pieces together suitably, establishes the relevant recovery.<sup>12</sup>

It is useful to point out that the torsion-freeness property of  $\Gamma_{abc}$  can be expressed as the configuration

$$\Gamma_{(ABC)A'B'C'} = \Gamma_{(ABC)(A'B')C'} = \Gamma_{(ABC)C'(A'B')} + 2\Theta_{(ABC)(A'\gamma_{B'})C'}. \quad (2.47a)$$

Since the world  $\Gamma$ -structures of Eqs. (2.44b) and (2.44c) do not bear symmetry in the indices  $a$  and  $b$ , we can say that Eq. (2.44a) does not generally lead to the statement<sup>13</sup>

$$\gamma_{A'(ABC)} = 0. \quad (2.47b)$$

One can fix up the primed-index symmetry exhibited by the relations (2.47a) by making use of Eqs. (2.44).

The basic  $\gamma$ -formalism device for computing covariant derivatives of spin densities is taken as an affine quantity  $\gamma_a$  that arises out of the metric prescription [3]

$$\nabla_a \varepsilon_{BC} = 0 \Leftrightarrow \gamma_a - \gamma_{aB}{}^B = 0. \quad (2.48)$$

Consequently,  $\gamma_a$  behaves under changes of coordinates in  $\mathfrak{M}$  as a covariant vector. It thus occurs in the formal configuration

$$\nabla_a \gamma_{BC} = \nabla_a (\gamma \varepsilon_{BC}) = \varepsilon_{BC} \nabla_a \gamma, \quad (2.49a)$$

and likewise enters into the expansion

$$\nabla_a \gamma = \partial_a \gamma - \gamma \gamma_a, \quad (2.49b)$$

which constitutes the prototype in the  $\gamma$ -formalism for covariant derivatives of complex spin-scalar densities of weight +1. Evidently, the right-hand side of

<sup>12</sup>As  $g_{bc} \partial_a g^{bc} = (-2) \Gamma_a$ , one can assert that Eq. (2.46a) amounts to nothing else but a spinor version of the classical relation  $\partial_a g_{bc} = 2\Gamma_{a(bc)}$ .

<sup>13</sup>Equation (2.47b) gives rise to decomposable spin connexions which describe the affine geometry of the class of conformally flat spacetimes referred to in Section 1.

Eq. (2.49b) stands for a covariant expansion for the independent component of  $\gamma_{AB}$ . For the density (2.22), we then have (see, e.g., Refs. [3, 6])

$$\nabla_a \alpha = \partial_a \alpha - \mathfrak{w} \alpha \gamma_a. \quad (2.50)$$

Needless to say, the computational device that arises from

$$\nabla_a \varepsilon_{B'C'} = 0 \Leftrightarrow \bar{\gamma}_a - \gamma_{aB'}{}^{B'} = 0, \quad (2.51)$$

is appropriate for the case that involves the complex conjugates of spin-scalar densities. When differentiating covariantly spin-scalar densities that bear both weights and antiweights, we must therefore utilize devices which are prescribed as suitable linear combinations of  $\gamma_a$  and  $\bar{\gamma}_a$ . For the density (2.23a), for instance, we have

$$\nabla_a \beta = \partial_a \beta - \beta (\mathfrak{a} \gamma_a + \mathfrak{b} \bar{\gamma}_a). \quad (2.52)$$

If  $\beta$  carries an absolute weight according to Eq. (2.23b), we will get

$$\nabla_a \beta = \partial_a \beta - 2\mathfrak{a} \beta \operatorname{Re} \gamma_a, \quad (2.53a)$$

that is to say,

$$\nabla_a \beta = \partial_a \beta + 2\mathfrak{a} \beta \theta_a. \quad (2.53b)$$

Hence, the combination of the definition (2.43) with the expansion

$$\nabla_a |\gamma| = \partial_a |\gamma| + |\gamma| \theta_a, \quad (2.53c)$$

shows that  $|\gamma|$  bears covariant constancy in the  $\gamma$ -formalism. The affine device for the composite spin-scalar density (2.24) is thus prescribed as

$$\nabla_a \Omega = \partial_a \Omega - \Omega (\mathfrak{a} \gamma_a + \mathfrak{b} \bar{\gamma}_a + \mathfrak{c} \operatorname{Re} \gamma_a). \quad (2.54a)$$

As an interesting example, we have

$$\begin{aligned} \nabla_a [\exp(i\Phi)] &= \partial_a [\exp(i\Phi)] - \exp(i\Phi) (\gamma_a - \operatorname{Re} \gamma_a) \\ &= i \exp(i\Phi) (\partial_a \Phi - \operatorname{Im} \gamma_a) \\ &= i \exp(i\Phi) (\partial_a \Phi + 2\Phi_a). \end{aligned} \quad (2.54b)$$

Covariant differentials of arbitrary spin-tensor densities can be specified by invoking the outer-product prescriptions given previously. For instance, setting

$$U_{BC\dots D} \doteq \beta T_{BC\dots D}, \quad (2.55)$$

with  $T_{BC\dots D}$  being some spin tensor, yields the expansion

$$\nabla_a U_{B\dots} = \partial_a U_{B\dots} - \gamma_{aB}{}^M U_{M\dots} - \dots - (\mathfrak{a} \gamma_a + \mathfrak{b} \bar{\gamma}_a) U_{B\dots}. \quad (2.56)$$

The covariant derivative of  $\Sigma_{aAA'}$ , say, is thus written down as

$$\nabla_a \Sigma_{bBB'} = \partial_a \Sigma_{bBB'} - \Gamma_{ab}{}^c \Sigma_{cBB'} - (\gamma_{aB}{}^M \Sigma_{bMB'} + \text{c.c.}) - \theta_a \Sigma_{bBB'}. \quad (2.57)$$

When combined with Eqs. (2.13a), the property

$$\nabla_a |\gamma| = 0 \quad (2.58)$$

then enables us to state that the derivative (2.57) vanishes. Therefore, Eqs. (2.13a) and (2.36) imply that all the  $\Sigma$ -connecting objects must bear covariant constancy in the  $\gamma$ -formalism too.

A glance at Eqs. (2.30) and (2.32) tells us that the rules for writing covariant derivatives of spin tensors in both formalisms are symbolically the same, but a corresponding spin-affine connexion  $\Gamma_{aB}^C$  and its complex conjugate should take over the computational role within the  $\varepsilon$ -framework. Thus, for an Hermitian world-spin tensor  $\kappa_{BB'}^b$ , we must have the  $\varepsilon$ -formalism expansion

$$\nabla_a \kappa_{BB'}^b = \partial_a \kappa_{BB'}^b + \Gamma_{ac}^b \kappa_{BB'}^c - (\Gamma_{aB}^C \kappa_{CB'}^b + \text{c.c.}), \quad (2.59a)$$

which is manifestly invariant under the world-covariant change

$$\Gamma_{aB}^C \mapsto \Gamma_{aB}^C + i\iota_a \varepsilon_B^C, \quad (2.59b)$$

with  $\text{Re}(i\iota_a) = 0$ . A suggestible procedure for building up  $\Gamma_{aB}^C$  consists in implementing the relationships (2.3) and taking the limit as  $\gamma$  tends to 1. Putting it into practice would nonetheless entail the annihilation of  $\theta_a$ , whence the numbers of independent components of  $\Gamma_{ab}^c$  and  $\Gamma_{aB}^C$  would have to be regularized from the beginning. Accordingly, we must necessarily take up the contracted prescription

$$-\text{Re} \Gamma_{aB}^B = \Pi_a, \quad (2.60)$$

whence the overall expression for  $\Gamma_{aB}^B$  has to be written as

$$\Gamma_{aB}^B = -(\Pi_a + 2i\varphi_a), \quad (2.61)$$

with  $\Pi_a$  and  $\varphi_a$  being world vectors [3].

Owing to the spin-density character of the  $\varepsilon$ -formalism connecting objects, no metric meaning can be assigned to  $\Pi_a$ . When considered together with Eq. (2.43), this fact constitutes one of the structural differences between the formalisms. The quantities  $\Phi_a$  and  $\varphi_a$  enter into the schemes as affine electromagnetic potentials that fulfill the Weyl gauge principle, in addition to satisfying wave equations which have the same form. In the presence of electromagnetic curvatures, the imaginary part of Eq. (2.42) may be utilized in the limiting case for making up  $\Gamma_{aB}^B$  symbolically. When the limiting procedure is implemented in the absence of fields,  $\Phi_a$  turns out to vanish in some gauge frame. We will describe the limiting process at greater length later upon deriving the eigenvalue equations and specifying the gauge behaviours of typical spin-affine structures.

The right-hand side of the tensor relation (2.21) is also proportional to  $\tau \varepsilon_{AB}$ , with  $\tau$  amounting to a complex spin-scalar density of weight +1 given as  $\gamma\eta$ . Thus, we can write the expansion

$$\begin{aligned} \nabla_a (\tau \varepsilon_{BC}) &= \partial_a (\tau \varepsilon_{BC}) - \tau \Gamma_{aD}^D \varepsilon_{BC} \\ &= (\partial_a \tau - \tau \Gamma_{aD}^D) \varepsilon_{BC} = (\nabla_a \tau) \varepsilon_{BC}, \end{aligned} \quad (2.62)$$

which leads us to stating that the set of affine computational devices for the  $\varepsilon$ -formalism can be entirely obtained in any gauge frame from that for the  $\gamma$ -formalism by making the simultaneous replacements

$$\theta_a \rightarrow \Pi_a, \quad \Phi_a \rightarrow \varphi_a. \quad (2.63)$$

It is evident that Eq. (2.60) emerges directly from

$$\nabla_a(\varepsilon_{BC}\varepsilon_{B'C'}) = 0, \quad (2.64)$$

whilst Eq. (2.33) appears as the vanishing derivative

$$\nabla_a \Sigma_{BB'}^b = \partial_a \Sigma_{BB'}^b + \Gamma_{ac}^b \Sigma_{BB'}^c - (\Gamma_{aB}^C \Sigma_{CB'}^b + \text{c.c.}) - \Pi_a \Sigma_{BB'}^b. \quad (2.65a)$$

It follows that the  $\varepsilon$ -formalism counterpart of Eq. (2.34c) is given by

$$\Gamma_{bc}^a n^c = \Sigma_{AA'}^a (\Gamma_{bB}^A \zeta^B \zeta^{A'} + \text{c.c.}) - \zeta^A \zeta^{A'} \partial_b \Sigma_{AA'}^a + \Pi_b n^a. \quad (2.65b)$$

The recovery in the  $\varepsilon$ -formalism of covariant-derivative patterns for arbitrary world tensors may be achieved from the equivalent requirements [3]

$$\nabla_a u^b = \Sigma_{BB'}^b \nabla_a u^{BB'}, \quad (2.66a)$$

and

$$\nabla_a u^{BB'} = \Sigma_b^{BB'} \nabla_a u^b, \quad (2.66b)$$

where  $u^b$  amounts to a world vector and  $u^{BB'}$  is an Hermitian spin-tensor density of absolute weight +1. Some manipulations involving rearrangements of index configurations then yield the affine relationship [23]

$$\Gamma_{AA'BB'CC'} + \Sigma_{hCC'} \partial_{AA'} \Sigma_{BB'}^h = (\Gamma_{AA'BC} \varepsilon_{B'C'} + \text{c.c.}) + \Pi_{AA'} \varepsilon_{BC} \varepsilon_{B'C'}, \quad (2.67a)$$

where

$$\Gamma_{AA'BB'CC'} = \Sigma_{AA'}^a \Sigma_{BB'}^b \Sigma_{CC'}^c \Gamma_{abc}, \quad (2.67b)$$

and

$$\partial_{AA'} = \Sigma_{AA'}^a \partial_a. \quad (2.67c)$$

Equations (2.67) exhibit the world covariance of  $\Gamma_{aBC}$  and its complex conjugate. The piece  $\Gamma_{A(BC)A'(B'C')}$  and the spinor decomposition of  $\Gamma_{a[bc]}$  arising here are both of the same form as the ones given by Eqs. (2.44). Also, the expression (2.47a) for the torsionlessness of  $\Gamma_{abc}$  still holds formally, but the traceful part of  $\Gamma_{a(bc)}$  is now subject to<sup>14</sup>

$$\Gamma_a + \Sigma_h^{BB'} \partial_a \Sigma_{BB'}^h = 0. \quad (2.68)$$

Transvecting Eq. (2.67a) with  $\varepsilon^{BC} \varepsilon^{B'C'}$  easily establishes the appropriateness of the condition (2.68). Likewise, recalling Eqs. (2.13a) and (2.43) brings back the  $\gamma$ -formalism equality

$$\begin{aligned} & \sigma_{AA'}^a \sigma_{BB'}^b \sigma_{CC'}^c \Gamma_{abc} + |\gamma| \sigma_{hCC'} \partial_{AA'} (|\gamma|^{-1} \sigma_{BB'}^h) \\ &= (\gamma_{AA'BC} \gamma_{B'C'} + \text{c.c.}) + \theta_{AA'} \gamma_{BC} \gamma_{B'C'}, \end{aligned} \quad (2.69)$$

provided that

$$\sigma_{AA'}^a \sigma_{BB'}^b \sigma_{CC'}^c \Gamma_{abc} = |\gamma|^3 \Sigma_{AA'}^a \Sigma_{BB'}^b \Sigma_{CC'}^c \Gamma_{abc}. \quad (2.70)$$

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<sup>14</sup>The vanishing of the derivative (2.57) reproduces the condition (2.68).

## 2.4 Eigenvalue Equations and Metric Expressions

One of the most significant features of the formalisms is the covariant constancy of the  $\varepsilon$ -metric spinors. This allows the implementation of the  $\gamma$ -formalism statement

$$\nabla_a \gamma_{BC} = (\gamma^{-1} \nabla_a \gamma) \gamma_{BC}, \quad (2.71)$$

which, when combined with Eq. (2.49b), yields the expansion

$$\nabla_a \gamma_{BC} = (\partial_a \log \gamma - \gamma_a) \gamma_{BC}. \quad (2.72)$$

Equations (2.1) and (2.2) then give the coupled eigenvalue equations

$$\nabla_a \gamma_{BC} = i(\partial_a \Phi + 2\Phi_a) \gamma_{BC}, \quad (2.73a)$$

and

$$\nabla_a \gamma^{BC} = (-i)(\partial_a \Phi + 2\Phi_a) \gamma^{BC}, \quad (2.73b)$$

along with their complex conjugates [3, 11]. The expansion (2.72) is consistent with Eqs. (2.54) and (2.58) as can be seen by working out the right-hand side of Eq. (2.71). We thus have

$$\gamma^{-1} \nabla_a \gamma = \frac{1}{2} \gamma^{BC} \nabla_a \gamma_{BC} = \exp(-i\Phi) \nabla_a \exp(i\Phi). \quad (2.74)$$

It becomes evident that the occurrence of a purely imaginary eigenvalue in Eqs. (2.73) is associated to a property of the  $\gamma$ -formalism which had been exhibited before by the conditions (2.36) and (2.37).

The partial derivative carried implicitly by the left-hand side of Eq. (2.71) can be isolated by utilizing the outer-product device

$$\theta_a \gamma_{BC} = (i\partial_a \Phi) \gamma_{BC} - \partial_a \gamma_{BC}, \quad (2.75a)$$

which comes from the computational prescription

$$\begin{aligned} \theta_a \gamma_{BC} &= \gamma_{BC} \partial_a \log[\gamma^{-1} \exp(i\Phi)] \\ &= \gamma(\partial_a \gamma^{-1}) \gamma_{BC} + (i\partial_a \Phi) \gamma_{BC} \\ &= \gamma[\partial_a(\gamma^{-1} \gamma_{BC}) - \gamma^{-1} \partial_a \gamma_{BC}] + (i\partial_a \Phi) \gamma_{BC}. \end{aligned} \quad (2.75b)$$

Thus, part of the information contained in  $\partial_a \gamma_{BC}$  gets annihilated by the information carried by  $\theta_a \gamma_{BC}$  when we bring together the pieces of  $\nabla_a \gamma_{BC}$ . This procedure gives rise to the following equations:

$$\partial_a \gamma_{BC} = (i\partial_a \Phi - \theta_a) \gamma_{BC}, \quad (2.76a)$$

$$\partial_a \gamma^{BC} = (\theta_a - i\partial_a \Phi) \gamma^{BC}, \quad (2.76b)$$

and

$$\partial_a (\gamma_{BC} \gamma_{B'C'}) = -2\theta_a \gamma_{BC} \gamma_{B'C'}, \quad (2.77a)$$

$$\partial_a (\gamma^{BC} \gamma^{B'C'}) = 2\theta_a \gamma^{BC} \gamma^{B'C'}. \quad (2.77b)$$

The eigenvalue carried by Eq. (2.76a) equals  $\partial_a \log \gamma$  whence the parts of the contracted spin affinity (2.42) can be expressed as

$$\theta_a = \frac{1}{2} \text{Re}[\gamma^{BC} (\nabla_a - \partial_a) \gamma_{BC}], \quad (2.78a)$$

and<sup>15</sup>

$$2\Phi_a = \frac{1}{2} \text{Im}[\gamma^{BC}(\nabla_a - \partial_a)\gamma_{BC}]. \quad (2.78b)$$

A system of covariant eigenvalue equations for the non-Hermitian  $\sigma$ -objects arises from Eqs. (2.33) and (2.73). For bringing out the pattern of a typical eigenvalue, it suffices to derive the equation for either element of any of the conjugate pairs

$$\{(\sigma_{bB}^{A'}, \Sigma_{bB}^{A'}), (\sigma_{bB'}^A, \Sigma_{bB'}^A)\}.$$

Thus, taking account of the prescription, say,

$$\nabla_a \sigma_{bB}^{A'} = \sigma_b^{AA'} \nabla_a \gamma_{AB}, \quad (2.79)$$

and employing the expansion

$$\nabla_a \sigma_{bB}^{A'} = \Sigma_{bB}^{A'} \nabla_a \exp(i\Phi) + \exp(i\Phi) \nabla_a \Sigma_{bB}^{A'}, \quad (2.80)$$

yields

$$\nabla_a \sigma_{bB}^{A'} = i(\partial_a \Phi + 2\Phi_a) \sigma_{bB}^{A'}. \quad (2.81)$$

It should be noticed that Eqs. (2.80) and (2.81) imply that

$$\nabla_a \Sigma_{bB}^{A'} = 0, \quad (2.82)$$

in agreement with the covariant constancy of the  $\Sigma$ -objects.

If  $\gamma$  is taken as a covariantly constant quantity in the  $\gamma$ -formalism, we may recover the expression (2.43) from Eq. (2.71), and achieve a metric specification of  $\Phi_a$  that enhances the absence of geometric electromagnetic fields, namely,

$$(-2)\Phi_a = \partial_a \Phi = \nabla_a \Phi. \quad (2.83)$$

To characterize this situation in an invariant way, it is enough to implement the condition

$$\nabla_a \gamma_{BC} = 0, \quad (2.84)$$

which evidently produces a commutativity property involving the action of the metric spinors for the  $\gamma$ -formalism and the action of the pertinent  $\nabla$ -operator. Equation (2.84) appears as a necessary and sufficient condition for the non-Hermitian  $\sigma$ -objects to bear covariant constancy. A procedure for illustrating the above statements, amounts to letting  $\partial_a$  act on the matrix configuration for  $\gamma_{BC}$ , likewise making use of a matrix form of Eq. (2.84). In effect, we have

$$\begin{pmatrix} 0 & \partial_a \gamma \\ -\partial_a \gamma & 0 \end{pmatrix} = \begin{pmatrix} 0 & \gamma \gamma_{aB}^B \\ -\gamma \gamma_{aB}^B & 0 \end{pmatrix}, \quad (2.85a)$$

whence, in view of Eq. (2.49b), we are led to

$$\gamma_a = \partial_a \log \gamma \Leftrightarrow \nabla_a \gamma = 0. \quad (2.85b)$$

Equations (2.85b) can be alternatively derived by partially differentiating the relations (2.3). We then obtain the intermediate-stage configurations

$$\gamma \partial_a \gamma_{BC} - (\partial_a \gamma) \gamma_{BC} = \gamma^2 \partial_a (\gamma^{-1} \gamma_{BC}) = 0, \quad (2.86a)$$

---

<sup>15</sup>It can be established from Eqs. (2.78) that the world covariance of  $\gamma_{aB}^C$  rests upon the world invariance of the  $\gamma$ -metric spinors.

which exhibit the gauge-invariant property<sup>16</sup>

$$\partial_a \varepsilon_{BC} = 0. \quad (2.86b)$$

It follows that Eq. (2.76a) can be reset as

$$\partial_a \log \gamma = \frac{1}{2} \gamma^{BC} \partial_a \gamma_{BC}. \quad (2.86c)$$

Apparently, the only procedure for extracting the spacetime information encoded into  $\gamma_{aB}{}^B$  is associated to the implementation of the affine prescription (2.45c). With regard to this observation, the key idea is to introduce the definition [23, 28]

$$\partial_a \log \mu = \sigma_h^{BB'} \partial_a \sigma_{BB'}^h, \quad (2.87)$$

with  $\mu$  standing for a mixed real-scalar density of world weight  $-1$  and absolute weight  $+4$ . Hence, recalling Eq. (2.43) yields the expressions

$$|\gamma|^4 = \mu(-\mathfrak{g})^{1/2}, \quad (2.88a)$$

and

$$(-4)\theta_a = \partial_a \log[\mu(-\mathfrak{g})^{1/2}]. \quad (2.88b)$$

It must be observed that the world-spin character of the derivative carried by the right-hand side of Eq. (2.87) can be clearly determined by contracting with  $\sigma_b^{BB'}$  the configuration<sup>17</sup>

$$\partial_a \sigma_{BB'}^b = (\gamma_{aB}{}^C \sigma_{CB'}^b + \text{c.c.}) - \Gamma_{ac}{}^b \sigma_{BB'}^c, \quad (2.89)$$

which arises from Eq. (2.35), and likewise reinstates the relation (2.45c). If use is made of Eq. (2.58), we may infer that  $\mu$  has to satisfy the condition

$$\nabla_a \mu = 0. \quad (2.90)$$

In case the limit as the pair  $(|\gamma|, \Phi)$  tends to  $(1, 0)$  is carried out, the eigenvalues borne by Eqs. (2.73) will turn out to equal  $\pm 2i\Phi_a$ . Consequently, because of the covariant constancy of the  $\varepsilon$ -metric spinors, the behaviour of the left-hand sides of those equations can be controlled by the expansion (2.49b). As provided by the eigenvalue equations (2.76), the description of the limiting process is based on the gauge-invariant  $\partial$ -constancy of the  $\varepsilon$ -metric spinors, which implies that both eigenvalues tend to zero when the limit is actually implemented. Taking up covariantly constant  $\gamma$ -metric spinors would thus make  $\Phi_a$  into a vanishing gradient, and the outcome of the limit of  $\gamma_{aB}{}^B$  would appear as a useless quantity. Therefore, if  $\Phi_a$  is taken as a gradient, we will have to reconstruct the contracted affine structures for the  $\varepsilon$ -formalism apart from the ones for the  $\gamma$ -formalism, but  $\varphi_a$  will have to carry a gradient character as well insofar as any shift from one formalism to the other must not produce any electromagnetic fields at all. If the  $\gamma$ -metric spinors are taken to have non-vanishing covariant derivatives, then the form of the imaginary part of  $\gamma_{aB}{}^B$  will be left unchanged, but we shall still have to take account of Eq. (2.60) in order to recover  $\Gamma_{aB}{}^B$ . We will elaborate further upon this situation in the following Subsection.

<sup>16</sup>We should stress that the operator  $\partial_a$  bears gauge invariance since arbitrary coordinates on  $\mathfrak{M}$  do not carry any spin character at all. Because of this fact, we will in Subsection 2.5 indiscriminately implement the relation  $\partial'_a = \partial_a$ .

<sup>17</sup>Equation (2.87) is compatible with the standard world-affine transformation laws.

## 2.5 Generalized Gauge Transformation Laws

As covariant derivatives generally possess the same form in both formalisms, the gauge behaviours of  $\gamma_{aB}^C$  and  $\Gamma_{aB}^C$  can be specified from one another by just replacing kernel letters. The original procedure for describing these behaviours [3], amounts in either case to taking up the covariance requirement

$$\nabla'_a \xi'_B = \Lambda_B^C \nabla_a \xi_C, \quad (2.91)$$

with  $\xi_A$  being an arbitrary spin vector. Hence, by writing out the expansions of Eq. (2.91) explicitly, and using the derivative device

$$\Lambda_B^C \partial_a \xi_C = \partial'_a \xi'_B - (\partial_a \Lambda_B^C) \xi_C, \quad (2.92)$$

after invoking the arbitrariness of  $\xi_A$ , we end up with the configuration

$$\vartheta'_{aB}{}^D \Lambda_D^C = \Lambda_B^D \vartheta_{aD}^C + \partial_a \Lambda_B^C, \quad (2.93)$$

where the kernel letter  $\vartheta$  stands for either  $\gamma$  or  $\Gamma$ , as before. Obviously, either of the affinities occurring in Eq. (2.93) can be picked out by adequately coupling all the involved pieces with an inverse  $\Lambda$ -matrix. We have, for instance,

$$\vartheta'_{aB}{}^C = \Lambda_B^D \vartheta_{aD}^M \Lambda_M^{-1C} + (\partial_a \Lambda_B^M) \Lambda_M^{-1C}. \quad (2.94)$$

There is an alternative procedure for deriving the law (2.94) which appropriately mixes up the unprimed and primed gauge frames. This consists in applying the Leibniz rule to the requirement (2.91), thereby supposing that any gauge-matrix components can always be covariantly differentiated in the same way as ordinary spin tensors. One thus obtains the correlation

$$\nabla'_a \xi'_B = \nabla_a \xi'_B - (\nabla_a \Lambda_B^C) \xi_C, \quad (2.95)$$

which immediately yields Eq. (2.93).

The behaviour of any contracted spin-affine structure for either formalism can be particularly attained by working out the coordinate derivative of the definition (2.26). For this purpose, we first note that Eqs. (2.76) yield<sup>18</sup>

$$\partial_a (M^{AB} M_{CD}) = 0, \quad (2.96)$$

whence it is legitimate to account for the relation

$$2\partial_a \Delta_\Lambda = M^{AB} \partial_a (\Lambda_A^C \Lambda_B^D) M_{CD}. \quad (2.97)$$

Additionally, carrying out the expansion of the  $\partial$ -piece borne by the right-hand side of Eq. (2.97), and invoking the prescription (2.93), leads to the value

$$2\partial_a \Delta_\Lambda = U_a^{(M)} - V_a^{(M)}, \quad (2.98a)$$

which carries the contributions

$$U_a^{(M)} = M^{AB} (\vartheta'_{aA}{}^N \Lambda_N^C \Lambda_B^D + \vartheta'_{aB}{}^N \Lambda_A^C \Lambda_N^D) M_{CD}, \quad (2.98b)$$

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<sup>18</sup>We recall here that the kernel letter  $M$  presumably denotes either  $\gamma$  or  $\varepsilon$ .

and

$$\begin{aligned}
V_a^{(M)} &= M^{AB}(\Lambda_A^N \Lambda_B^D \vartheta_{aN}^C + \Lambda_A^C \Lambda_B^N \vartheta_{aN}^D) M_{CD} \\
&= M^{AB}(\Lambda_A^N \Lambda_B^C \vartheta_{a[NC]} - \Lambda_A^C \Lambda_B^N \vartheta_{a[NC]}) \\
&= 2M^{AB} \Lambda_A^C \Lambda_B^D \vartheta_{a[CD]}.
\end{aligned} \tag{2.98c}$$

For the  $\gamma$ -formalism, we use Eqs. (2.18) to perform the computations

$$\begin{aligned}
U_a^{(\gamma)} &= \gamma^{AB}(\gamma'_{aA}{}^M \gamma'_{MB} + \gamma'_{aB}{}^M \gamma'_{AM}) \\
&= 2\Delta_\Lambda \gamma'^{AB} \gamma'_{a[AB]} = 2\Delta_\Lambda \gamma'_{aB}{}^B,
\end{aligned} \tag{2.99a}$$

and

$$V_a^{(\gamma)} = \gamma^{AB} \gamma_{aC}{}^C \gamma'_{AB} = 2\Delta_\Lambda \gamma_{aB}{}^B. \tag{2.99b}$$

In a similar way, for the  $\varepsilon$ -formalism, we utilize Eqs. (2.25) to obtain

$$\begin{aligned}
U_a^{(\varepsilon)} &= \varepsilon^{AB}(\Gamma'_{aA}{}^M \Lambda_M^C \Lambda_B^D + \Gamma'_{aB}{}^M \Lambda_A^C \Lambda_M^D) \varepsilon_{CD} \\
&= \Delta_\Lambda \varepsilon^{AB}(\Gamma'_{aA}{}^M \varepsilon'_{MB} + \Gamma'_{aB}{}^M \varepsilon'_{AM}) \\
&= 2\Delta_\Lambda \Gamma'_{aB}{}^B,
\end{aligned} \tag{2.100a}$$

and

$$V_a^{(\varepsilon)} = 2\varepsilon^{AB} \Lambda_A^C \Lambda_B^D \Gamma_{a[CD]} = 2\Delta_\Lambda \Gamma_{aB}{}^B. \tag{2.100b}$$

It follows that

$$\partial_a \Delta_\Lambda = \Delta_\Lambda (\vartheta'_{aB}{}^B - \vartheta_{aB}{}^B), \tag{2.101a}$$

whence Eq. (2.94) can be cast into the form

$$\vartheta'_{aB}{}^C = \vartheta_{aB}{}^C + \frac{1}{2}(\partial_a \log \Delta_\Lambda) \delta_B^C. \tag{2.101b}$$

Then, making a contraction over the indices  $B$  and  $C$  carried by Eq. (2.101b), gives rise to the laws<sup>19</sup>

$$\gamma'_{aB}{}^B = \gamma_{aB}{}^B + \partial_a \log \Delta_\Lambda, \tag{2.102a}$$

and

$$\Gamma'_{aB}{}^B = \Gamma_{aB}{}^B + \partial_a \log \Delta_\Lambda, \tag{2.102b}$$

together with their complex conjugates.

From Eqs. (2.102), we see that the gauge behaviours of the structures (2.42) and (2.61) have to be specified by

$$A'_a = A_a - \partial_a \Lambda, \tag{2.103a}$$

$$\theta'_a = \theta_a - \partial_a \log \rho, \tag{2.103b}$$

and

$$\Pi'_a = \Pi_a - \partial_a \log \rho, \tag{2.104}$$

with the quantity  $A_a$  amounting to either  $\Phi_a$  or  $\varphi_a$ . We point out that the transformation law for  $|\gamma|$  as given in Subsection 2.2 can be recovered out of

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<sup>19</sup>One should bear in mind that the metric prescriptions for lowering and raising spinor indices in both formalisms must strictly involve quantities defined in the same gauge frames.

combining Eqs. (2.43) and (2.103b). By appealing to Eqs. (2.76), we can also achieve the description of the geometric character of  $\exp(i\Phi)$  from

$$\partial'_a \Phi' = \partial_a \Phi + 2\partial_a \Lambda. \quad (2.105)$$

It turns out that the gauge behaviour of the partial derivatives of the  $\gamma$ -metric spinors can be fully described by the law

$$\partial'_a \log \gamma' = \partial_a \log \gamma + \partial_a \log \Delta_\Lambda. \quad (2.106)$$

One thus concludes that the eigenvalues of Eqs. (2.73) bear gauge invariance [3, 11], whence the invariant character of Eq. (2.84) can be fixed by taking into consideration the  $\gamma$ -formalism prescription

$$\nabla'_a \gamma'_{BC} = \Delta_\Lambda \nabla_a \gamma_{BC}. \quad (2.107)$$

The establishment of the law (2.103a) definitively characterizes  $\Phi_a$  and  $\varphi_a$  as the electromagnetic potentials of  $\gamma_{aB}{}^C$  and  $\Gamma_{aB}{}^C$ , respectively. Likewise, Eq. (2.107) shows that if the  $\gamma$ -metric spinors are taken to bear covariant constancy in the unprimed frame, they will have to be looked upon as covariantly constant entities in the primed frame as well. Hence, if  $\Phi_a$  is a gradient in the unprimed frame, it will also be a gradient in any other frame. Consequently, as was observed before, taking the limit as  $\gamma$  tends to 1 would annihilate both parts of  $\gamma_{aB}{}^B$  in the unprimed frame. In such circumstances, the primed-frame pieces  $\Phi'_a$  and  $\theta'_a$  would evidently become proportional to  $\partial_a \Lambda$  and  $\partial_a \log \rho$ , whence any contracted affine structures for the  $\varepsilon$ -formalism would have to be entirely reconstructed in accordance with the prescriptions (2.61) and (2.104). It should be clear that the gauge behaviours of  $\partial\gamma$ -equations like (2.76) and (2.77) may be controlled in any case by Eq. (2.106). Therefore, one can allow for a metric principle which describes in a gauge-invariant fashion the geometric aspect of the  $\gamma$ -formalism that concerns the presence or absence of electromagnetic curvatures.

We can covariantly keep track of gauge behaviours by assuming that any  $\nabla$ -derivative of some spin tensor or density can be carried out in any frame regardless of whether the kernel letter of the object to be differentiated is primed or unprimed. Let us, in effect, consider the  $\gamma$ -formalism expansion

$$\nabla_a \gamma'_{BC} = \partial_a \gamma'_{BC} - \gamma_{aM}{}^M \gamma'_{BC}. \quad (2.108)$$

Interchanging the roles of the frames and making use of Eq. (2.102a) yields

$$\nabla'_a \gamma_{BC} = \nabla_a \gamma_{BC} - (\partial_a \log \Delta_\Lambda) \gamma_{BC}, \quad (2.109)$$

whence the covariant derivative of Eq. (2.108) obeys the relation

$$\nabla_a \gamma'_{BC} = \nabla'_a \gamma'_{BC} + (\partial_a \log \Delta_\Lambda) \gamma'_{BC}. \quad (2.110)$$

As a consequence of Eq. (2.110), we can account for the contracted derivatives

$$\gamma^{BC} \nabla'_a \gamma_{BC} = \gamma^{BC} \nabla_a \gamma_{BC} - \partial_a \log(\Delta_\Lambda)^2, \quad (2.111a)$$

and

$$\gamma'^{BC} \nabla_a \gamma'_{BC} = \gamma'^{BC} \nabla'_a \gamma'_{BC} + \partial_a \log(\Delta_\Lambda)^2, \quad (2.111b)$$

which clearly reflect the interchange of frames implemented above. It appears that if either of Eqs. (2.111) had been considered alone, then the gauge-frame prescription for the other could have been obtained by effecting the substitution

$$\Delta_\Lambda \mapsto \delta_\Lambda. \quad (2.112)$$

Now, by taking account of Eq. (2.109), we write down the expansions

$$\begin{aligned} \nabla'_a \gamma'_{BC} &= \nabla'_a (\Delta_\Lambda \gamma_{BC}) = \Delta_\Lambda \nabla'_a \gamma_{BC} + (\nabla'_a \Delta_\Lambda) \gamma_{BC} \\ &= \Delta_\Lambda \nabla_a \gamma_{BC} + (\nabla'_a \Delta_\Lambda - \partial'_a \Delta_\Lambda) \gamma_{BC}, \end{aligned} \quad (2.113)$$

which suggest ascribing a gauge-scalar character to  $\Delta_\Lambda$ , namely,

$$\nabla'_a \Delta_\Lambda = \partial'_a \Delta_\Lambda = \partial_a \Delta_\Lambda = \nabla_a \Delta_\Lambda. \quad (2.114)$$

From Eqs. (2.113), it also follows that<sup>20</sup>

$$\gamma'^{BC} \nabla'_a \gamma'_{BC} = \gamma^{BC} \nabla_a \gamma_{BC}, \quad (2.115)$$

whence the condition (2.36) is subject to the homogeneous law

$$\nabla'_a (\gamma'_{BC} \gamma'_{B'C'}) = |\Delta_\Lambda|^2 \nabla_a (\gamma_{BC} \gamma_{B'C'}). \quad (2.116)$$

A covariant mixed-frame property arises when we work out covariant derivatives of the unprimed-index  $\gamma$ -metric spinors for the primed frame. For instance, taking Eqs. (2.114) into account gives

$$\nabla_a \gamma'_{BC} = \nabla_a (\Delta_\Lambda \gamma_{BC}) = \Delta_\Lambda \nabla_a \gamma_{BC} + (\partial_a \Delta_\Lambda) \gamma_{BC}, \quad (2.117a)$$

whence, because of Eqs. (2.108)-(2.110), we can write

$$\nabla'_a (\Delta_\Lambda \gamma_{BC}) + (\partial_a \Delta_\Lambda) \gamma_{BC} = \Delta_\Lambda \nabla'_a \gamma_{BC} + (2\partial_a \Delta_\Lambda) \gamma_{BC}. \quad (2.117b)$$

Equation (2.109) then yields the prescription

$$\delta_\Lambda \nabla_a \gamma'_{BC} = \nabla'_a \gamma_{BC} + 2(\partial_a \log \Delta_\Lambda) \gamma_{BC}, \quad (2.118)$$

which upon transvection with  $\gamma'^{BC}$  leads to

$$\gamma'^{BC} \nabla'_a \gamma_{BC} = \gamma'^{BC} \nabla_a \gamma'_{BC} - \partial_a \log(\Delta_\Lambda)^4. \quad (2.119)$$

Therefore, the sum of contracted  $\nabla$ -derivatives having the same gauge-frame mixing is maintained when we interchange the frames, namely,

$$\gamma'^{BC} \nabla_a \gamma'_{BC} + \gamma'_{BC} \nabla_a \gamma'^{BC} = \gamma^{BC} \nabla'_a \gamma_{BC} + \gamma_{BC} \nabla'_a \gamma'^{BC}. \quad (2.120)$$

An important feature of the covariant-derivative prescriptions we have exhibited is that they can be used as a metric technique whereby one may describe subtly the transformation laws for the contracted spin affinities of the  $\gamma$ -formalism. The best way of examining this situation is to observe that a requirement of the form of Eq. (2.91) comes out when we insert into the relation (2.110) the expansion

$$\nabla_a \gamma'_{BC} = \Lambda_B^L \Lambda_C^M \nabla_a \gamma_{LM} + \nabla_a (\Lambda_B^L \Lambda_C^M) \gamma_{LM}. \quad (2.121)$$

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<sup>20</sup>Equations (2.114) enable one to say that the functions  $\rho$  and  $\Lambda$  carried by the definition (2.14a) are world-spin scalars.

Hence, implementing Eqs. (2.114) in the form

$$\nabla_a(\Lambda_B^L \Lambda_C^M) \gamma_{LM} = (\partial_a \Delta_\Lambda) \gamma_{BC}, \quad (2.122)$$

produces the statement

$$\nabla'_a \gamma'_{BC} = \Lambda_B^L \Lambda_C^M \nabla_a \gamma_{LM}, \quad (2.123)$$

which effectively recovers the laws (2.102a) and (2.107).

In both gauge frames, there occurs annihilation of part of the information carried by the covariant derivatives of  $\Lambda_B^L \Lambda_C^M$  when the overall differential expansions are appropriately contracted with  $\gamma_{LM}$  or  $\gamma'_{LM}$ . The amount of information annihilated in each frame is not gauge invariant, and can be actually calculated by performing the relevant expansion. What results is, in effect, that the pieces

$$(\Delta_\Lambda \gamma_{aM}^M \gamma_{BC}, \Delta_\Lambda \gamma'_{aM}^M \gamma'_{BC}) \quad (2.124)$$

cancel out when the contracted derivatives are individually built up. To establish this statement, it is enough to rewrite Eq. (2.110) as

$$\nabla'_a \gamma'_{BC} = \nabla_a \gamma'_{BC} - \nabla_a(\Lambda_B^L \Lambda_C^M) \gamma_{LM}, \quad (2.125a)$$

or, more explicitly, as<sup>21</sup>

$$\nabla_a(\Lambda_B^L \Lambda_C^M) \gamma_{LM} = \partial_a(\Lambda_B^L \Lambda_C^M) \gamma_{LM}. \quad (2.125b)$$

Particularly, the pieces occurring in the configuration

$$\gamma^{BC} \nabla_a(\Lambda_B^L \Lambda_C^M) \gamma_{LM} = \gamma^{BC} \partial_a(\Lambda_B^L \Lambda_C^M) \gamma_{LM}, \quad (2.126)$$

carry only gauge-invariant information.

At this point, it is expedient to reexpress Eq. (2.94) as

$$\gamma'_{aBC} = \Lambda_B^L \Lambda_C^M \gamma_{aLM} + (\partial_a \Lambda_B^L) \Lambda_C^M \gamma_{LM}. \quad (2.127)$$

Because of the pattern of Eq. (2.14c), we can write the useful relation

$$(\partial_a \Delta_\Lambda) \gamma_{BC} = 2(\partial_a \Lambda_B^L) \Lambda_C^M \gamma_{LM}, \quad (2.128)$$

whence

$$\gamma'_{aBC} = \Lambda_B^L \Lambda_C^M \gamma_{aLM} + \frac{1}{2}(\partial_a \Delta_\Lambda) \gamma_{BC}, \quad (2.129)$$

which recovers the law (2.101b). Hence, multiplying Eq. (2.129) by  $\gamma'^{BC}$  reinstates the law (2.102a), since

$$\gamma'^{BC} \Lambda_B^L \Lambda_C^M \gamma_{aLM} = \delta_\Lambda \Delta_\Lambda \gamma'^{BC} \gamma_{a[BC]} = \gamma_{aB}^B, \quad (2.130a)$$

and

$$\frac{1}{2} \gamma'^{BC} (\partial_a \Delta_\Lambda) \gamma_{BC} = \delta_\Lambda \partial_a \Delta_\Lambda = \partial_a \log \Delta_\Lambda. \quad (2.130b)$$

Furthermore, if we implement the splittings

$$\gamma'_{aBC} = \gamma'_{a(BC)} + \frac{1}{2} \gamma'_{aM}^M \gamma'_{BC}, \quad (2.131a)$$

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<sup>21</sup>We notice that Eqs. (2.125) recover the relation (2.122).

and

$$\Lambda_B^L \Lambda_C^M \gamma_{aLM} = \Lambda_B^L \Lambda_C^M \gamma_{a(LM)} + \frac{1}{2} \Delta_\Lambda \gamma_{aM}^M \gamma_{BC}, \quad (2.131b)$$

we will obtain the spin-tensor prescription

$$\gamma'_{a(BC)} = \Lambda_B^L \Lambda_C^M \gamma_{a(LM)} = \Delta_\Lambda \gamma_{a(BC)}, \quad (2.132)$$

along with the law

$$\gamma'_{aBC} = \Lambda_B^L \Lambda_C^M \gamma_{a(LM)} + \frac{1}{2} \Delta_\Lambda (\gamma_{aM}^M + \partial_a \log \Delta_\Lambda) \gamma_{BC}. \quad (2.133)$$

Upon proceeding to the derivation of the transformation laws for the  $\varepsilon$ -formalism, we must call upon the structure (2.94) and work out the primed-frame configuration

$$\Gamma'_{aBC} = \Gamma'_{aB}{}^M \varepsilon'_{MC}. \quad (2.134)$$

The relations (2.114) and (2.128) still remain valid as they stand there since both formalisms involve one and the same gauge group, but the law (2.133) has to be replaced with

$$\Gamma'_{aBC} = (\Delta_\Lambda)^{-1} \Lambda_B^L \Lambda_C^M \Gamma_{a(LM)} + \frac{1}{2} (\Gamma_{aM}^M + \partial_a \log \Delta_\Lambda) \varepsilon_{BC}. \quad (2.135)$$

Equations (2.101b) and (2.102b) are thereby recovered, and we can write the prescription

$$\Gamma'_{a(BC)} = (\Delta_\Lambda)^{-1} \Lambda_B^L \Lambda_C^M \Gamma_{a(LM)} = \Gamma_{a(BC)}, \quad (2.136)$$

whence  $\Gamma_{a(BC)}$  is an invariant spin-tensor density of weight  $-1$ . It can therefore be said that the symmetric parts of any spin-affine connexions for both formalisms carry a gauge-covariant character. By making use of Eqs. (2.133) and (2.135) along with the equality

$$\partial_a |\Delta_\Lambda|^2 = 2 \operatorname{Re}(\bar{\Delta}_\Lambda \partial_a \Delta_\Lambda), \quad (2.137)$$

we also establish that the relationships (2.44a) and (2.67a) behave covariantly.

Worthy of special consideration is the fact that covariant differentials in both formalisms of any typical geometric objects carry the same gauge characters as the differentiated objects themselves. This property exhibits the existence of a formal analogy between covariant derivatives of world and spin quantities in  $\mathfrak{M}$ . It just comes from the combination of the outer-product extension of the requirement (2.91) with the prescriptions for building up arbitrary spin-tensor densities. For example, the gauge behaviour of the expansion (2.56) is specified by

$$\nabla'_a U'_{BC...D} = (\Delta_\Lambda)^a (\bar{\Delta}_\Lambda)^b \Lambda_B^L \Lambda_C^M \dots \Lambda_D^N \nabla_a U_{LM...N}. \quad (2.138)$$

The prescription (2.54b) thus undergoes the transformation

$$\nabla'_a \exp(i\Phi') = \Delta_\Lambda |\Delta_\Lambda|^{-1} \nabla_a \exp(i\Phi), \quad (2.139)$$

while  $\nabla_a S_{AA'}^b$  behaves as

$$\nabla'_a \sigma_{AA'}^b = |\Delta_\Lambda| \nabla_a \sigma_{AA'}^b, \quad (2.140a)$$

and

$$\nabla'_a \Sigma_{AA'}^b = \nabla_a \Sigma_{AA'}^b. \quad (2.140b)$$

Equations (2.140) establish the gauge invariance of the  $\nabla$ -constancy property of the elements of the set (2.8).

### 3 Spin Curvature and Wave Equations

We shall now describe systematically the curvature spinors of  $\gamma_{aB}{}^C$  and  $\Gamma_{aB}{}^C$ . The pertinent computational devices carry the definition of a set of spinor differential operators that constitute bivector configurations for torsionless covariant-derivative commutators. A rough form of such operators was first utilized by Penrose and Rindler [8] for describing the propagation of some charged spinning fields in the presence of external electromagnetic fields. Upon working out the procedures that yield the wave equations for gravitons, one has necessarily to implement a version of the gravitational Bianchi identity which amounts to an extension of that borne by the spinor classification schemes brought up earlier. As before, we will bring out the role of the geometric structures for the  $\gamma$ -formalism without leaving out the characterization of their  $\varepsilon$ -formalism counterparts.

A particularly remarkable geometric feature of the formalisms is that whereas any curvature spinors for the  $\gamma$ -formalism are subject to tensorial gauge transformation laws, the corresponding structures for the  $\varepsilon$ -formalism carry a gauge-invariant density character. In either formalism, any conjugate gravitational and electromagnetic wave functions supply dynamical states for gravitons and geometric photons of opposite handednesses. The gravitational pieces of any curvature splittings for both formalisms likewise give rise to a common gauge-invariant cosmological constant. Wave functions for gravitons are geometrically expressed in the same way as for the physically weak cases of covariantly constant  $\gamma$ -metric spinors, but wave functions for geometric photons are in any such case automatically made into useless vanishing quantities. Indeed, a system of gauge-covariant field and wave equations bearing prescribed index configurations is what controls the propagation of gravitons and photons in  $\mathfrak{M}$  [24].

The relevant commutators along with the curvature spinors are constructed in Subsection 3.1. We will exhibit the electromagnetic field and wave equations in Subsection 3.2. The gravitational statements are given afterwards in Subsection 3.3. As regards the curvature structures themselves, any wave functions shall be taken as classical fields from the physical viewpoint. The inclusion of the description of Dirac fields in  $\mathfrak{M}$  is made in Subsection 3.4. Either of the potentials of Eq. (2.103a) will be denoted simply as  $\Phi_a$ . All the main procedures shall obviously be completed in the presence of electromagnetic curvatures. The traces of any world quantities of valences  $\{0, 2\}$  and  $\{2, 0\}$  will be denoted by the kernel letters used to write the aforesaid quantities. Our choice of sign convention for the Ricci tensor  $R_{ab}$  of  $\Gamma_{abc}$  coincides with the one made in Ref. [8], namely,

$$R_{ab} \doteq R_{ahb}{}^h,$$

with  $R_{abc}{}^d$  being the corresponding Riemann tensor.

#### 3.1 Commutators and Curvature Spinors

In either formalism, the information on the respective curvature splitting is carried by the world-covariant commutator [3]

$$[\nabla_a, \nabla_b] S^{cDD'} \doteq 2\nabla_{[a}(\nabla_{b]} S^{cDD'}) = 0, \quad (3.1)$$

where  $S^{cDD'}$  is one of the entries of the set (2.8). Expanding the middle configuration of Eq. (3.1), and invoking the covariant-differential prescriptions of

Subsection 2.3, yields the relation<sup>22</sup>

$$(S^{cAB'} W_{abA}{}^B + \text{c.c.}) + S^{hBB'} R_{abh}{}^c = 0, \quad (3.2)$$

with

$$W_{abA}{}^B = 2\partial_{[a}\vartheta_{b]A}{}^B - (\vartheta_{aA}{}^C \vartheta_{bC}{}^B - \vartheta_{bA}{}^C \vartheta_{aC}{}^B) \quad (3.3)$$

being the defining expression for a typical Infeld-van der Waerden mixed curvature object for either formalism. Likewise, transvecting Eq. (3.2) with  $S_{cDB'}$  gives

$$2W_{abA}{}^B + \delta_A{}^B W_{abA'}{}^{A'} = S_{AB'}^c S^{dBB'} R_{abcd}, \quad (3.4)$$

whence we can state that

$$4 \operatorname{Re} W_{abC}{}^C = R_{abh}{}^h \equiv 0. \quad (3.5)$$

Evidently, the procedure that yields Eq. (3.5) brings about annihilation of the information carried by  $R_{abcd}$ , whence the trace  $W_{abC}{}^C$  appears as a purely imaginary quantity in either formalism. The simplest manner of deriving the explicit spin-affine expressions for the conjugate  $W$ -traces of both formalisms, is to contract the free spinor indices of Eq. (3.3), verifying thereafter that the contracted pattern for the involved quadratic  $\vartheta$ -piece vanishes identically. We thus obtain the electromagnetic expression

$$W_{abC}{}^C = 2\partial_{[a}\vartheta_{b]C}{}^C = (-4i)\partial_{[a}\Phi_{b]}. \quad (3.6)$$

The  $W$ -objects for both formalisms can be alternatively obtained out of the commutator

$$S_{AA'}^a S_{BB'}^b [\nabla_a, \nabla_b] \zeta^C = W_{AA'BB'M}{}^C \zeta^M, \quad (3.7)$$

where  $\zeta^C$  is some spin vector. One can likewise recover the expression (3.3) from Eq. (3.7) by replacing in the case of either formalism  $\zeta^C$  with a spin quantity defined as the outer product of a gauge-invariant world vector with a suitable Hermitian  $S$ -matrix (see Ref. [33] and Section 4).

The gravitational contribution to the curvature structure of either formalism amounts to

$$W_{ab(AB)} = \frac{1}{2} S_{AB'}^c S_B^{dB'} R_{abcd}, \quad (3.8)$$

which bears the symmetries exhibited by Eqs. (2.9). We have then been led to the world-spin curvature splitting

$$W_{abAB} = \frac{1}{2} S_{AB'}^c S_B^{dB'} R_{abcd} - iF_{ab} M_{AB}, \quad (3.9)$$

with  $F_{ab}$  being the Maxwell tensor

$$F_{ab} \doteq 2\partial_{[a}\Phi_{b]} = 2\nabla_{[a}\Phi_{b]}. \quad (3.10)$$

A symmetrization over the indices  $A$  and  $B$  of Eq. (3.9) obviously causes annihilation of the electromagnetic information carried by  $W_{abAB}$ . In the  $\gamma$ -formalism, we have the covariant prescription

$$W'_{abAB} = \Lambda_A{}^C \Lambda_B{}^D W_{abCD} = \Delta_\Lambda W_{abAB}. \quad (3.11)$$

---

<sup>22</sup>The  $\varepsilon$ -formalism version of Eq. (3.1) carries a term proportional to  $\partial_{[a}\Pi_{b]}$  which may also be taken to vanish. This point will be made clear in Section 4.

In the  $\varepsilon$ -formalism, the symmetric pieces  $W_{ab(AB)}$  and  $W_{ab(A'B')}$  are, respectively, taken as invariant spin-tensor densities of weight  $-1$  and antiweight  $-1$ , whence we have the law

$$W'_{abAB} = (\Delta_\Lambda)^{-1} \Lambda_A^C \Lambda_B^D W_{abCD} = W_{abAB}, \quad (3.12)$$

along with the complex conjugates of the prescriptions (3.11) and (3.12). We should stress that  $W_{abA}^B$  and  $W_{abC}^C$  are gauge-invariant tensors in both formalisms.

The overall curvature spinors of either  $\gamma_{aB}^C$  or  $\Gamma_{aB}^C$  arise [23, 24, 33] from the bivector configuration borne by Eq. (3.9). We have, in effect,

$$S_{AA'}^a S_{BB'}^b W_{abCD} = M_{A'B'} \omega_{ABCD} + M_{AB} \omega_{A'B'CD}, \quad (3.13)$$

where

$$\omega_{ABCD} = \omega_{(AB)CD} \doteq \frac{1}{2} S_{AA'}^a S_B^{bA'} W_{abCD}, \quad (3.14a)$$

and

$$\omega_{A'B'CD} = \omega_{(A'B')CD} \doteq \frac{1}{2} S_{AA'}^a S_{B'}^{bA} W_{abCD}. \quad (3.14b)$$

Due to the gauge characters of the  $W$ -objects, the curvature spinors for the  $\gamma$ -formalism are subject to the tensor laws

$$\omega'_{ABCD} = \Lambda_A^L \Lambda_B^M \Lambda_C^R \Lambda_D^S \omega_{LMRS} = (\Delta_\Lambda)^2 \omega_{ABCD}, \quad (3.15a)$$

and

$$\omega'_{A'B'CD} = \Lambda_{A'}^{L'} \Lambda_{B'}^{M'} \Lambda_C^R \Lambda_D^S \omega_{L'M'RS} = |\Delta_\Lambda|^2 \omega_{A'B'CD}, \quad (3.15b)$$

whereas the ones for the  $\varepsilon$ -formalism are invariant spin-tensor densities prescribed by

$$\omega'_{ABCD} = (\Delta_\Lambda)^{-2} \Lambda_A^L \Lambda_B^M \Lambda_C^R \Lambda_D^S \omega_{LMRS} = \omega_{ABCD}, \quad (3.16a)$$

and

$$\omega'_{A'B'CD} = |\Delta_\Lambda|^{-2} \Lambda_{A'}^{L'} \Lambda_{B'}^{M'} \Lambda_C^R \Lambda_D^S \omega_{L'M'RS} = \omega_{A'B'CD}. \quad (3.16b)$$

The Riemann-Christoffel curvature structure of  $\mathfrak{M}$  can be completely recovered from the pair

$$\mathbf{G} = (\omega_{AB(CD)}, \omega_{A'B'(CD)}). \quad (3.17)$$

Thus, the elements of the  $\mathbf{G}$ -pair for each formalism enter into the corresponding spinor expression for  $R_{abcd}$  according to the gauge-covariant Hermitian prescription

$$R_{AA'BB'CC'DD'} = (M_{A'B'} M_{C'D'} \omega_{AB(CD)} + M_{AB} M_{C'D'} \omega_{A'B'(CD)}) + \text{c.c.} \quad (3.18)$$

This property was established [23] out of utilizing the expansion (3.13) along with the formulae

$$S_{CA'}^c S_D^{dA'} S_{[c}^{EB'} S_{d]B'}^F = (-2) M^{(E}{}_C M^{F)}{}_D, \quad (3.19a)$$

and

$$S_{CA'}^c S_D^{dA'} S_{[c}^{BE'} S_{d]B}^{F'} = M^{E'F'} M_{(CD)} \equiv 0, \quad (3.19b)$$

for reexpressing the right-hand side of Eq. (3.8) as

$$\frac{1}{2}S_{CA'}^c S_D^{dA'} R_{abcd} = S_{A'[a}^A S_{b]}^{BA'} \omega_{AB(CD)} + S_{A[a}^{A'} S_{b]}^{B'A} \omega_{A'B'(CD)}. \quad (3.20)$$

We can see that the above-mentioned procedure recovers the symmetries borne by the bivector expansion (3.13). It properly annihilates the entire complex-conjugate piece of Eq. (3.18), and thereby allows one to pick up from  $R_{abcd}$  the elements of the pair given as Eq. (3.17). Hence, the gravitational curvature spinors of either formalism are defined as the entries of the respective **G**-pair. The symmetries carried by the configuration (3.20) correspond to the skew symmetry in the indices of the pairs  $ab$  and  $cd$  borne by  $R_{abcd}$ . In view of the spacetime symmetry  $R_{abcd} = R_{cdab}$ , we also have to demand the index-pair symmetries

$$\omega_{AB(CD)} = \omega_{(CD)AB}, \quad \omega_{A'B'(CD)} = \omega_{(CD)A'B'}. \quad (3.21)$$

The second entry of the **G**-pair for either formalism has therefore to be regarded as an Hermitian entity. Since unprimed and primed spinor indices always assume algebraically independent values [8], it can be said that there is no fixed prescription for ordering the indices of  $\omega_{A'B'(CD)}$ .

The cosmological interpretation of the gravitational spinors given in Ref. [8] may be attained if Eq. (3.18) is rewritten as

$$R_{AA'BB'CC'DD'} = (M_{A'B'} M_{C'D'} X_{ABCD} + M_{AB} M_{C'D'} \Xi_{CA'DB'}) + \text{c.c.}, \quad (3.22)$$

with the  $X\Xi$ -spinors being defined by

$$X_{ABCD} \doteq \frac{1}{4} M^{A'B'} M^{C'D'} R_{AA'BB'CC'DD'} = \omega_{AB(CD)}, \quad (3.23a)$$

and<sup>23</sup>

$$\Xi_{CA'DB'} \doteq \frac{1}{4} M^{AB} M^{C'D'} R_{AA'BB'CC'DD'} = \omega_{A'B'(CD)}. \quad (3.23b)$$

As was pointed up in Section 1, the developments leading to this insight had supported a spinor translation of Einstein's equations [8, 15]. We note that one of Eqs. (3.21) leads us to the statement

$$M^{AD} X_{A(BC)D} = 0 \Leftrightarrow M^{BC} X_{(A|BC|D)} = 0, \quad (3.24)$$

which produces the relations

$$M^{AD} X_{ABCD} = \chi M_{BC}, \quad (3.25a)$$

$$M^{BC} X_{ABCD} = \chi M_{AD}, \quad (3.25b)$$

and

$$X_{AB}{}^{AB} = 2\chi, \quad (3.25c)$$

with  $\chi$  obviously standing for a world-spin invariant. Therefore, we can write the first-left dual pattern [8]

$$^* R_{AA'BB'CC'DD'} = [(-i)(M_{A'B'} M_{C'D'} X_{ABCD} - M_{AB} M_{C'D'} \Xi_{CA'DB'})] + \text{c.c.} \quad (3.26)$$

---

<sup>23</sup>The world version of either  $\Xi$ -spinor enters the theoretical scheme as  $(-2)\Xi_{ab} \doteq \hat{s}R_{ab}$ , with  $\hat{s}$  being the operator involved in Eq. (2.44g).

Hence, calling for the world property  ${}^*R_{ab}{}^{bc} = 0$ , yields

$$M_{A'D'}M^{BC}X_{ABCD} = M_{AD}M^{B'C'}X_{A'B'C'D'}, \quad (3.27)$$

whence  $\text{Im } \chi = 0$ . The spinor  $\omega_{AB(CD)}$  thus possesses eleven real independent components while  $\omega_{A'B'(CD)}$  possesses nine [14], with the number of independent components of  $R_{abcd}$  being thereupon recovered in both formalisms. One can see from Eq. (3.22) that the spinor expression for the Ricci tensor appears as

$$R_{AA'BB'} = 2(\chi M_{AB}M_{A'B'} - \Xi_{AA'BB'}). \quad (3.28)$$

The  $\Xi$ -spinor of either formalism then satisfies the full Einstein's equations

$$2\Xi_{AA'BB'} = \kappa(T_{AA'BB'} - \frac{1}{4}TM_{AB}M_{A'B'}), \quad (3.29)$$

which correspond to

$$2\Xi_{ab} = \kappa \hat{s}T_{ab}, \quad (3.30)$$

where  $\kappa$  stands for the Einstein gravitational constant and  $T_{ab}$  amounts to the world version of the energy-momentum tensor of some sources. For the Ricci scalar, one accordingly has the general trace relation

$$R = 8\chi = 4\lambda + \kappa T, \quad (3.31)$$

with  $\lambda$  being the cosmological constant. It follows that, when only traceless sources are present, the spinor expression for the Einstein tensor<sup>24</sup> is given by

$$G_{AA'BB'} = -2\Xi_{AA'BB'} - \lambda M_{AB}M_{A'B'}. \quad (3.32)$$

The symmetries of  $X_{ABCD}$  considerably simplify the four-index reduction formula

$$\begin{aligned} X_{ABCD} = & X_{(ABCD)} - \frac{1}{4}(M_{AB}X^M{}_{(MCD)} + M_{AC}X^M{}_{(MBD)} + M_{AD}X^M{}_{(MBC)}) \\ & - \frac{1}{3}(M_{BC}X^M{}_{A(MD)} + M_{BD}X^M{}_{A(MC)}) - \frac{1}{2}M_{CD}X_{AB}{}^M{}_M. \end{aligned} \quad (3.33)$$

When combined together with Eqs. (3.23) and (3.25), this property affords us the relation<sup>25</sup>

$$X_{ABCD} = X_{(ABCD)} - \frac{2}{3}\chi M_{A(C}M_{D)B}, \quad (3.34)$$

with

$$X_{(ABCD)} = X_{A(BCD)} = X_{(ABC)D}. \quad (3.35)$$

Additionally, we stress that the Hermitian configuration

$$\begin{aligned} & (M_{A(C}M_{D)B}M_{A'B'}M_{C'D'}) + \text{c.c.} \\ & = M_{AD}M_{BC}M_{A'D'}M_{B'C'} - M_{AC}M_{BD}M_{A'C'}M_{B'D'}, \end{aligned} \quad (3.36)$$

<sup>24</sup>The Einstein tensor equals  $G_{ab} = \hat{\tau}R_{ab} \doteq R_{ab} - \frac{1}{2}Rg_{ab}$ , where  $\hat{\tau}$  is the so-called trace-reversal operator. This operator is linear and possesses the involutory property  $\hat{\tau}^2 = \text{identity}$ . It also commutes with  $\hat{s}$  (see, e.g., Ref. [8]).

<sup>25</sup>The quantity  $\Lambda$  used in Ref. [23] was imported from Penrose and Rindler [8]. It always obeys the relation  $\chi = 3\Lambda$ , but the equality  $\lambda = 6\Lambda$  holds only when  $T = 0$ .

gives rise to the splitting

$$\begin{aligned} & M_{A'B'}M_{C'D'}(X_{(ABCD)} - X_{ABCD}) + \text{c.c.} \\ &= \frac{2}{3}\chi(M_{AD}M_{BC}M_{A'D'}M_{B'C'} - M_{AC}M_{BD}M_{A'C'}M_{B'D'}). \end{aligned} \quad (3.37)$$

The electromagnetic contribution to the curvature spinors for either formalism comprises the pair of contracted pieces [24]

$$\mathbf{E} = (\omega_{ABC}{}^C, \omega_{A'B'C}{}^C), \quad (3.38a)$$

which enter into the bivector decomposition<sup>26</sup>

$$S_{AA'}^a S_{BB'}^b F_{ab} = \frac{i}{2}(M_{A'B'}\omega_{ABC}{}^C + M_{AB}\omega_{A'B'C}{}^C). \quad (3.38b)$$

From Eqs. (3.10), we get the relationships

$$\omega_{ABC}{}^C = 2i\nabla_{(A}^{C'}\Phi_{B)C'}, \quad (3.39a)$$

and

$$\omega_{A'B'C}{}^C = 2i\nabla_{(A'}^C\Phi_{B')C}, \quad (3.39b)$$

whence we are led to the spinor splittings

$$\omega_{ABCD} = \omega_{(AB)(CD)} + \frac{1}{2}\omega_{(AB)L}{}^L M_{CD}, \quad (3.40a)$$

and

$$\omega_{A'B'CD} = \omega_{(A'B')(CD)} + \frac{1}{2}\omega_{(A'B')L}{}^L M_{CD}, \quad (3.40b)$$

along with their complex conjugates. Whereas the electromagnetic pieces of Eqs. (3.40) behave in the  $\gamma$ -formalism as spin tensors, they occur in the  $\varepsilon$ -formalism as invariant spin-tensor densities subject to the laws

$$\omega'_{ABC}{}^C = (\Delta_\Lambda)^{-1}\Lambda_A{}^L\Lambda_B{}^M\omega_{LMC}{}^C = \omega_{ABC}{}^C, \quad (3.41a)$$

and

$$\omega'_{A'B'C}{}^C = (\bar{\Delta}_\Lambda)^{-1}\Lambda_{A'}{}^{L'}\Lambda_{B'}{}^{M'}\omega_{L'M'C}{}^C = \omega_{A'B'C}{}^C. \quad (3.41b)$$

As regards the computations that produce the derivation of any wave equations for either formalism, the key covariant-derivative pattern is written out explicitly as

$$[\nabla_{AA'}, \nabla_{BB'}] = M_{A'B'}\Delta_{AB} + M_{AB}\Delta_{A'B'}. \quad (3.42)$$

The  $\Delta$ -kernels involved on the right-hand side of Eq. (3.42) are both symmetric second-order differential operators which bear linearity as well as the Leibniz-rule property. In the  $\gamma$ -formalism, they behave formally under gauge transformations as covariant spin tensors, with the respective defining expressions being written as<sup>27</sup>

$$\Delta_{AB} = \nabla_{C'(A}\nabla_{B)}^{C'} - i\beta_{C'(A}\nabla_{B)}^{C'}, \quad (3.43)$$

<sup>26</sup>The spinors of Eq. (3.38a) obey the peculiar conjugacy relations  $\omega_{ABC}{}^C = -\omega_{ABC'}{}^{C'}$  and  $\omega_{A'B'C}{}^C = -\omega_{A'B'C'}{}^{C'}$ .

<sup>27</sup>Equation (3.43) can be reset as  $\Delta_{AB} = -\nabla_{(A}^{C'}\nabla_{B)C'}$ .

and

$$\Delta_{A'B'} = \nabla_{C(A'} \nabla_{B')}^C + i\beta_{C(A'} \nabla_{B')}^C, \quad (3.44)$$

where  $i\beta_a$  amounts to the eigenvalue carried by Eq. (2.73a). For the  $\varepsilon$ -formalism, we have

$$\Delta_{AB} = \nabla_{C'(A} \nabla_{B)}^{C'}, \quad \Delta_{A'B'} = \nabla_{C(A'} \nabla_{B')}^C, \quad (3.45)$$

with  $\Delta_{AB}$  and  $\Delta_{A'B'}$  thus behaving as invariant spin-tensor densities of weight  $-1$  and antiweight  $-1$ , respectively. It is useful to remark that the covariant constancy of  $M^{AB}M^{A'B'}$  enables one to define the contravariant form of any  $\Delta$ -operator. In particular, the  $\gamma$ -formalism version of  $\Delta^{AB}$ , for instance, appears as

$$\Delta^{AB} = -(\nabla^{C'(A} \nabla_{C'}^B) + i\beta^{C'(A} \nabla_{C'}^B), \quad (3.46a)$$

or, equivalently, as

$$\Delta^{AB} = \nabla_{C'}^{(A} \nabla^{B)C'}, \quad (3.46b)$$

with the relevant defining structure being in either formalism set as<sup>28</sup>

$$\Delta^{AB} \doteq M^{AC}M^{BD}\Delta_{CD} = M^{A(C}M^{D)B}\nabla_C^{M'}\nabla_{DM'}. \quad (3.47)$$

One of the implications of the eventual presence of electromagnetic pieces in curvature splittings is that an appropriate number of contributions carrying terms of the same type as the entries of Eq. (3.38a) must be borne by any  $\Delta$ -derivative of arbitrary outer-product configurations. Equations (3.7) and (3.42) thus suggest that some of the most elementary derivatives should be prescribed in either formalism as

$$\Delta_{AB}\zeta^C = \omega_{ABM}{}^C\zeta^M = X_{ABM}{}^C\zeta^M + \frac{1}{2}\omega_{ABM}{}^M\zeta^C, \quad (3.48a)$$

and

$$\Delta_{A'B'}\zeta^C = \omega_{A'B'M}{}^C\zeta^M = \Xi_{A'B'M}{}^C\zeta^M + \frac{1}{2}\omega_{A'B'M}{}^M\zeta^C. \quad (3.48b)$$

The basic prescriptions for computing  $\Delta$ -derivatives of a covariant spin vector  $\xi_A$  can be obtained from Eqs. (3.48) by carrying out Leibniz expansions of the product  $\zeta^C\xi_C$ . We then have<sup>29</sup>

$$\Delta_{AB}\xi_C = -\omega_{ABC}{}^M\xi_M = -(X_{ABC}{}^M\xi_M + \frac{1}{2}\omega_{ABM}{}^M\xi_C), \quad (3.49a)$$

and

$$\Delta_{A'B'}\xi_C = -\omega_{A'B'C}{}^M\xi_M = -(\Xi_{A'B'C}{}^M\xi_M + \frac{1}{2}\omega_{A'B'M}{}^M\xi_C), \quad (3.49b)$$

along with the complex conjugates of Eqs. (3.48) and (3.49). For the complex spin-scalar density defined by Eq. (2.22), we can write the derivatives

$$\Delta_{AB}\alpha = -\mathfrak{w}\alpha\omega_{ABC}{}^C, \quad (3.50a)$$

<sup>28</sup>Because of the symmetry of the  $\Delta$ -operators, there is no need for staggering their indices.

<sup>29</sup>When acting on a world-spin scalar  $h$ , the  $\Delta$ -operators recover the torsionlessness of  $\Gamma_{abc}$  as  $\Delta_{AB}h = 0$  and  $\Delta_{A'B'}h = 0$ .

and

$$\Delta_{A'B'}\alpha = -\mathfrak{w}\alpha\omega_{A'B'C}{}^C, \quad (3.50b)$$

which are usually thought of as coming from the integrability condition [23]

$$[\nabla_a, \nabla_b]\alpha = (-2\mathfrak{w}\alpha)\partial_{[a}\vartheta_{b]} = 2i\mathfrak{w}\alpha F_{ab}, \quad (3.51)$$

with  $\vartheta_a$  standing for either of the affine devices  $\gamma_a$  and  $\Gamma_{aB}{}^B$ . It is obvious that the right-hand sides of Eqs. (3.50) and (3.51) turn out to vanish when gradient potentials are allowed for. Because of the presupposition that both  $\partial_{[a}\theta_{b]}$  and  $\partial_{[a}\Pi_{b]}$  vanish, any real spin-scalar densities must behave in either formalism as numerical constants with respect to the action of the corresponding  $\Delta$ -operators. The patterns of  $\Delta$ -derivatives of some spin-tensor density can evidently be specified from Leibniz expansions like

$$\Delta_{AB}(\alpha B_{C...D}) = (\Delta_{AB}\alpha)B_{C...D} + \alpha\Delta_{AB}B_{C...D}, \quad (3.52)$$

with  $B_{C...D}$  being a spin tensor. It follows that if we invoke once again the outer-product extension of the requirement (2.91), observing that Eqs. (2.114) entail the constancy of  $\Delta_\Lambda$  with respect to the action of the commutator of Eq. (3.1), we shall conclude that the gauge behaviours of generic  $\Delta$ -derivatives bear both homogeneity and linearity in either formalism. For example, we have the  $\gamma$ -formalism law

$$\Delta'_{AB}(\alpha' B'_{C...D}) = (\Delta_\Lambda)\mathfrak{w}\Lambda_A{}^G\Lambda_B{}^H\Lambda_C{}^L\dots\Lambda_D{}^M\Delta_{GH}(\alpha B_{L...M}). \quad (3.53)$$

There are some situations of practical interest [24, 33] in which the calculation of  $\Delta$ -derivatives can be carried out as if electromagnetic pieces were absent from curvature splittings. The first point concerning this observation is related to the fact that there occurs a cancellation of those pieces whenever  $\Delta$ -derivatives of Hermitian quantities are explicitly computed in either formalism.<sup>30</sup> Such a cancellation likewise happens when we let  $\Delta$ -operators act freely upon spin tensors of valences  $\{a, a; 0, 0\}$  and  $\{0, 0; c, c\}$ . For  $\mathfrak{w} < 0$ , it still occurs in the expansion (3.52) when the valence of  $B_{C...D}$  equals  $\{0, -2\mathfrak{w}; 0, 0\}$  and  $\text{Im } \alpha \neq 0$  everywhere. A similar property also holds for cases that involve outer products between contravariant spin tensors and complex spin-scalar densities having suitable positive weights.

## 3.2 Wave Equations for Geometric Photons

In both formalisms, the wave functions for geometric photons constitute the bivector decomposition given by Eqs. (3.38). The relevant definitions are expressed as

$$\phi_{AB} \doteq \frac{i}{2}\omega_{ABC}{}^C, \quad \phi_{A'B'} \doteq \frac{i}{2}\omega_{A'B'C}{}^C, \quad (3.54)$$

together with the field-potential relationships

$$\phi_{AB} = -\nabla_{(A}^{C'}\Phi_{B)C'}, \quad \phi_{A'B'} = -\nabla_{(A'}^C\Phi_{B')C}, \quad (3.55a)$$

and

$$\phi^{AB} = \nabla_{C'}^{(A}\Phi^{B)C'}, \quad \phi^{A'B'} = \nabla_C^{(A'}\Phi^{B')C}. \quad (3.55b)$$

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<sup>30</sup>As an illustrative example, we have the pattern  $2\text{Re}(\sigma_{aB'}^A\Delta_{AB}u^{BB'}) = R_{ab}u^b$ .

These wave functions are inextricably rooted into the curvature structure of  $\mathfrak{M}$ , being locally considered as massless uncharged fields of spin  $\pm 1$ . At each point of  $\mathfrak{M}$ , they represent the six geometric degrees of freedom of  $W_{abA}{}^A$ , in accordance with the expansion

$$S_{AA'}^a S_{BB'}^b F_{ab} = M_{A'B'} \phi_{AB} + \text{c.c.}, \quad (3.56)$$

and its dual

$$S_{AA'}^a S_{BB'}^b F_{ab}^* = i(M_{AB} \phi_{A'B'} - \text{c.c.}). \quad (3.57)$$

In the  $\varepsilon$ -formalism,  $\phi_{AB}$  and  $\phi_{A'B'}$  bear gauge invariance, with any rearrangements of the indices carried by Eqs. (3.54) likewise leading to gauge-invariant fields. On the other hand, the only index configurations that yield invariant fields in the  $\gamma$ -formalism are supplied by  $\phi_A{}^B$  and  $\phi_{A'}{}^{B'}$ , which visibly carry an invariant spin-tensor character in the  $\varepsilon$ -formalism as well. The corresponding field equations arise from the coupled conjugate statements

$$\nabla^{AA'} (S_{AA'}^a S_{BB'}^b F_{ab} + i S_{AA'}^a S_{BB'}^b F_{ab}^*) = 0, \quad (3.58a)$$

and

$$\nabla^{AA'} (S_{AA'}^a S_{BB'}^b F_{ab} - i S_{AA'}^a S_{BB'}^b F_{ab}^*) = 0. \quad (3.58b)$$

We then have the Maxwell equations

$$\nabla^{AA'} (M_{A'B'} \phi_{AB}) = 0, \quad (3.59a)$$

and

$$\nabla^{AA'} (M_{AB} \phi_{A'B'}) = 0. \quad (3.59b)$$

In the  $\gamma$ -formalism, Eqs. (3.59) give rise to

$$\nabla^{AB'} \phi_{AB} = i \beta^{AB'} \phi_{AB}, \quad (3.60a)$$

$$\nabla_{AB'} \phi^{AB} = -i \beta_{AB'} \phi^{AB}, \quad (3.60b)$$

and

$$\nabla^{BA'} \phi_{A'B'} = -i \beta^{BA'} \phi_{A'B'}, \quad (3.60c)$$

$$\nabla_{BA'} \phi^{A'B'} = i \beta_{BA'} \phi^{A'B'}, \quad (3.60d)$$

with the  $\beta$ -spinor being the same as the one carried by the definition (3.43). The specification of the gauge behaviours of Eqs. (3.60) can be attained from the law

$$(\nabla'^{AB'} - i \beta'^{AB'}) \phi'_{AB} = \exp(2i\Lambda) (\nabla^{AB'} - i \beta^{AB'}) \phi_{AB}, \quad (3.61)$$

whence the gauge invariance of Maxwell's theory turns out to be exhibited by either

$$\nabla'^{AB'} \phi_A{}^B = |\Delta_\Lambda|^{-1} \nabla^{AB'} \phi_A{}^B = 0 \quad (3.62)$$

or the complex conjugates of Eqs. (3.62). Clearly, this result is compatible with the gauge invariance of the vacuum equations<sup>31</sup>

$$\nabla^a F_{ab} = 0, \quad \nabla^a F_{ab}^* = 0. \quad (3.63)$$

---

<sup>31</sup>The second of Eqs. (3.63) stands for the electromagnetic Bianchi identity.

In the  $\varepsilon$ -formalism, Eqs. (3.59) are reduced to the gauge-invariant massless-free-field equations

$$\nabla^{AB'}\phi_{AB} = 0, \quad \nabla^{BA'}\phi_{A'B'} = 0. \quad (3.64)$$

The gauge invariance of Eqs. (3.64) is independent of any choices of index configurations because of the  $\nabla$ -constancy of the  $\varepsilon$ -metric spinors.

In either formalism, the basic procedure for obtaining the wave equation that controls the propagation of  $\phi_A{}^B$ , amounts to operating on it with the splitting

$$\nabla_{A'}^C \nabla^{AA'} = \Delta^{AC} - \frac{1}{2} M^{AC} \square, \quad (3.65a)$$

and working out the resulting structure. For completing the calculational steps in a systematic fashion, it is necessary to take account of the algebraic rules

$$2\nabla_{[C}^{A'} \nabla_{A']} = M_{AC} \square = \nabla_D^{A'} (M_{CA} \nabla_{A'}^D), \quad (3.65b)$$

and

$$2\nabla_{A'}^{[C} \nabla^{A]} = M^{CA} \square = \nabla_{A'}^D (M^{AC} \nabla_D^{A'}), \quad (3.65c)$$

along with their complex conjugates and the gauge-invariant definition<sup>32</sup>

$$\square \doteq S_{MM'}^a S^{bMM'} \nabla_a \nabla_b = \nabla_{MM'} \nabla^{MM'}. \quad (3.65d)$$

In the  $\gamma$ -formalism, we thus have

$$\nabla_{A'}^C \nabla^{AA'} \phi_A{}^B = \Delta^{AC} \phi_A{}^B - \frac{1}{2} \gamma^{AC} \square \phi_A{}^B = 0. \quad (3.66)$$

Owing to the valence pattern of  $\phi_A{}^B$ , the  $\Delta$ -derivative of Eqs. (3.66) just carries  $X\phi$ -pieces, namely,

$$\Delta^{AC} \phi_A{}^B = X^{AC}{}_M{}^B \phi_A{}^M - X^{AC}{}_A{}^M \phi_M{}^B = \Delta^{A(B} \phi_A{}^{C)}. \quad (3.67)$$

The property concerning the symmetry brought out by Eqs. (3.67) can be deduced by allowing for the computation

$$\begin{aligned} & \Delta^{A[C} \phi_A{}^{B]} \\ &= X^{A[C}{}_{AM} \phi^{B]M} + X^{A[C}{}_{M}{}^{B]} \phi_A{}^M \\ &= \frac{1}{2} \gamma^{CB} X^A{}_{LMA} \phi^{LM} - X^{A[CB]M} \phi_{AM} \\ &= \frac{1}{2} \chi \gamma^{CB} (\gamma_{ML} \phi^{ML} - \gamma^{AM} \phi_{AM}) \equiv 0. \end{aligned} \quad (3.68)$$

Hence, by rearranging the indices of Eqs. (3.67) adequately, and invoking the expansion (3.34), we get the contribution

$$\Delta^{AB} \phi_A{}^C = \frac{4}{3} \chi \phi^{BC} - \omega^{(ABCD)} \phi_{AD}, \quad (3.69)$$

which leads us to the gauge-invariant equation

$$(\square + \frac{8}{3} \chi) \phi_A{}^B = (-2) \Psi_{AD}{}^{BC} \phi_C{}^D, \quad (3.70a)$$

---

<sup>32</sup>The property  $\nabla_a (M_{AB} M_{A'B'}) = 0$  implies that  $\nabla_{MM'} \nabla^{MM'} = \nabla^{MM'} \nabla_{MM'}$ .

with the definition

$$\Psi_{ABCD} \doteq \omega_{(ABCD)} = X_{(ABCD)}. \quad (3.70b)$$

For the reason that  $\phi_A^B$  bears a tensor character in both formalisms, one can say that the  $\varepsilon$ -formalism version of  $\Delta^{AC}\phi_A^B$  is formally the same as Eqs. (3.67). Then, the corresponding wave equation is an invariant tensor statement of the same form as Eq. (3.70a). The  $\varepsilon$ -formalism wave equation for  $\phi_{AB}$  may of course be readily written down as

$$(\square + \frac{8}{3}\chi)\phi_{AB} = 2\Psi_{AB}{}^{CD}\phi_{CD}. \quad (3.71)$$

These results agree with the fact that the wave function  $\phi_{AB}$  for the  $\varepsilon$ -formalism is a two-index covariant spin-tensor density of weight  $-1$ . Consequently, one may implement the purely gravitational pattern of Eqs. (3.67) upon expanding  $\Delta^{AB}\phi_{AC}$ . The  $\gamma$ -formalism version of Eq. (3.71) emerges from working out the configuration

$$2\Delta^{AC}\phi_{AB} - \gamma^{AC}\square\phi_{AB} = \nabla_{A'}^C(2i\beta^{AA'}\phi_{AB}), \quad (3.72a)$$

with the pertinent equation amounting, in effect, to the spin-tensor statement

$$(\square - 2i\beta^h\nabla_h - \Upsilon_{(\mathcal{P})} + \frac{8}{3}\chi)\phi_{AB} = 2\Psi_{AB}{}^{CD}\phi_{CD}, \quad (3.72b)$$

where

$$\Upsilon_{(\mathcal{P})} \doteq \beta^h\beta_h + i(\square\Phi + 2\nabla_h\Phi^h). \quad (3.72c)$$

Explicit calculations [23] show that the right-hand side of Eq. (3.72a) is essentially constituted by the Leibniz contributions

$$\beta^{AA'}\nabla_{CA'}\phi_{AB} = (\beta^h\nabla_h - \frac{1}{2}i\beta^h\beta_h)\phi_{BC}, \quad (3.72d)$$

and

$$(\nabla_{CA'}\beta^{AA'})\phi_{AB} = (\frac{1}{2}\square\Phi + \nabla_h\Phi^h)\phi_{BC} + 2\phi_C^A\phi_{AB}. \quad (3.72e)$$

Upon joining pieces together, we see that the (skew) quadratic term  $4i\phi_C^A\phi_{AB}$  cancels out because of the expansion

$$2\Delta^{AC}\phi_{AB} = \frac{8}{3}\chi\phi_B^C - 2\Psi_B{}^{CMN}\phi_{MN} - 2\omega^{AC}{}_M{}^M\phi_{AB}. \quad (3.72f)$$

In either formalism, the wave equation for  $\Phi_{AA'}$  can be derived by working out any of the relationships (3.55). For instance,

$$(-2)\phi_A^B = \nabla^{BB'}\Phi_{AB'} + M^{BC}\nabla_A^{B'}\Phi_{CB'}, \quad (3.73a)$$

whence

$$\nabla^{AA'}\nabla^{BB'}\Phi_{AB'} + \nabla^{AA'}(M^{BC}\nabla_A^{B'}\Phi_{CB'}) = 0. \quad (3.73b)$$

For the first piece of Eq. (3.73b), we may utilize the operator splitting

$$\nabla^{AA'}\nabla^{BB'} = \nabla^{BA'}\nabla^{AB'} + M^{AB}(\frac{1}{2}M^{A'B'}\square + \nabla_C^{(A'}\nabla^{B')C}), \quad (3.74)$$

to obtain the expression

$$\nabla^{AA'} \nabla^{BB'} \Phi_{AB'} = M^{AB} \left( \frac{1}{2} M^{A'B'} \square + \nabla_C^{(A'} \nabla^{B')C} \right) \Phi_{AB'} + \nabla^{BA'} \Theta, \quad (3.75a)$$

where  $\Theta$  is the Lorentz world scalar<sup>33</sup>

$$\Theta \doteq S_{MM'}^a S^{bMM'} \nabla_a \Phi_b = \nabla_{MM'} \Phi^{MM'}. \quad (3.75b)$$

For the other piece of Eq. (3.73b), we have the calculation

$$\begin{aligned} \nabla^{AA'} (M^{BC} \nabla_A^{B'} \Phi_{CB'}) &= \nabla^{AA'} (M^{BC} \nabla_{(A}^{B'} \Phi_{C)B'} + \frac{1}{2} M^{BC} M_{CA} \Theta) \\ &= -\frac{1}{2} \nabla^{BA'} \Theta, \end{aligned} \quad (3.76)$$

with Eqs. (3.62) having been employed.

The complex conjugates of Eqs. (3.46) supply the  $\gamma$ -formalism configuration

$$\nabla^{AA'} \nabla^{BB'} \Phi_{AB'} = \gamma^{AB} \left( \frac{1}{2} \gamma^{A'B'} \square \Phi_{AB'} + \Delta^{A'B'} \Phi_{AB'} \right) + \nabla^{BA'} \Theta, \quad (3.77)$$

whence adding together Eqs. (3.76) and (3.77) produces the structure

$$\nabla_B^{A'} \Theta - \gamma^{A'B'} \square \Phi_{BB'} - 2 \Delta^{A'B'} \Phi_{BB'} = 0. \quad (3.78)$$

By virtue of the Hermiticity of  $\Phi_{AB'}$ , the  $\Delta$ -expansion of Eq. (3.78) carries only the gravitational contribution borne by

$$\Delta^{A'B'} \Phi_{AB'} = \frac{1}{2} R_A^{A'BB'} \Phi_{BB'}, \quad (3.79)$$

with  $R_{AA'BB'}$  being given by the relation (3.28). Some trivial manipulations then yield the statements

$$\square \Phi_{AA'} + R_{AA'}^{BB'} \Phi_{BB'} - \nabla_{AA'} \Theta = 0, \quad (3.80)$$

and

$$\left( \square + \frac{R}{4} \right) \Phi_{AA'} - \nabla_{AA'} \Theta = 0, \quad (3.81)$$

which turn out to be equivalent whenever  $\Xi_{ab} = 0$ .

It has become obvious that the  $\varepsilon$ -formalism version of  $\Delta^{A'B'} \Phi_{AB'}$  bears the same form as the structure (3.79). Combining Eqs. (3.75) and (3.76) thus leads to a wave equation of the same form as the statement (3.81). Since the action of either  $\square$ -operator on any appropriate Hermitian  $S$ -matrix produces a vanishing outcome, we can establish that electromagnetic potentials for both formalisms must coincide with each other when electromagnetic curvatures are present. If instead of Eq. (3.73a) we had used the configuration for either  $\phi_{AB}$  or  $\phi^{AB}$ , we would have derived the same wave equations for  $\Phi_{AA'}$  as the ones exhibited above. In either formalism, the pattern of the spacetime wave equation for  $\Phi_a$  could therefore be recovered from Eq. (3.80) by invoking the requirement (2.33). We stress that the main point regarding the situation at issue is associated to a commonness feature of the Maxwell bivectors carried by the formalisms. It apparently gets strengthened when one carries out the world computation

$$\begin{aligned} \nabla^b F_{ba} &= \nabla^b (\nabla_b \Phi_a - \nabla_a \Phi_b) = \square \Phi_a - g^{bh} ([\nabla_h, \nabla_a] + \nabla_a \nabla_h) \Phi_b \\ &= \square \Phi_a - [\nabla_b, \nabla_a] \Phi^b - \nabla_a \Theta = \square \Phi_a + R_a^b \Phi_b - \nabla_a \Theta. \end{aligned} \quad (3.82)$$

---

<sup>33</sup>The quantity  $\Theta$  transforms under the action of the Weyl group as  $\Theta' = \Theta - \square \Lambda$ .

### 3.3 Wave Equations for Gravitons

The totally symmetric spinors borne by Eqs. (3.70) and (3.71) are the Weyl spinor fields of the formalisms. In both frameworks, they enter together with their complex conjugates into the spinor expression for the Weyl tensor  $C_{abcd}$  of  $\mathfrak{M}$ , according to the scheme [8, 15]

$$S_{AA'}^a S_{BB'}^b S_{CC'}^c S_{DD'}^d C_{abcd} = M_{A'B'} M_{C'D'} \Psi_{ABCD} + \text{c.c.} \quad (3.83)$$

At each point of  $\mathfrak{M}$ , the conjugate  $\Psi$ -fields for either formalism are taken to represent the ten independent degrees of freedom of  $g_{ab}$ . Physically, they are massless uncharged wave functions carrying spin  $\pm 2$ , which are deeply involved in the gravitational structure of  $\mathfrak{M}$ . To derive the relevant field equations, one has to utilize the expression (3.26) for working out the coupled Bianchi relations<sup>34</sup>

$$M^{C'D'} \nabla^{AA'} R_{AA'BB'CC'DD'} = 0, \quad (3.84a)$$

and

$$M^{CD} \nabla^{AA'} R_{AA'BB'CC'DD'} = 0. \quad (3.84b)$$

In the  $\gamma$ -formalism, Eq. (3.84a) takes the explicit form

$$\nabla_{B'}^A X_{ABCD} - 2i\beta_{B'}^A X_{ABCD} = \nabla_B^{A'} \Xi_{A'B'CD}, \quad (3.85)$$

which can be reset as

$$\nabla^{AA'} (X_{ABC}{}^D \gamma_{A'B'}) = \nabla^{AA'} (\Xi_{A'B'C}{}^D \gamma_{AB}). \quad (3.86)$$

Hence, performing a symmetrization over the indices  $B$ ,  $C$  and  $D$  of Eq. (3.85), and recalling the property (3.35), yields the statement

$$\nabla_{B'}^A \Psi_{ABCD} - 2i\beta_{B'}^A \Psi_{ABCD} = \nabla_{(B}^{A'} \Xi_{CD)A'B'}. \quad (3.87)$$

We emphasize that the skew parts in  $B$  and  $C$  of the individual terms of Eq. (3.85), produce a differential cosmological relationship whose applicability does not depend upon whether electromagnetic curvatures are present or absent. We have, in effect,

$$\nabla_{B'}^A X_{A[BC]D} - 2i\beta_{B'}^A X_{A[BC]D} = \nabla_{[B}^{A'} \Xi_{C]DA'B'}, \quad (3.88)$$

whence, after performing some calculations, we obtain

$$(-8) \nabla^{AA'} \Xi_{AA'BB'} = \nabla_{BB'} R. \quad (3.89)$$

The procedure that leads to the statement (3.87), annihilates the information carried by the  $\chi$ -piece of Eq. (3.34). In vacuum, we can then write the gauge-covariant eigenvalue equations

$$\nabla^{AB'} \Psi_{ABCD} = 2i\beta^{AB'} \Psi_{ABCD}, \quad (3.90a)$$

and

$$\nabla_{AB'} \Psi^{ABCD} = (-2i)\beta_{AB'} \Psi^{ABCD}, \quad (3.90b)$$

---

<sup>34</sup>The world version of the gravitational Bianchi identity is written as  $\nabla^a R_{abcd} = 0$ .

which can be rewritten as the invariant massless-free-field equation

$$\nabla^{AA'} \Psi_{AB}{}^{CD} = 0. \quad (3.91)$$

The  $\varepsilon$ -formalism version of  $\Psi_{AB}{}^{CD}$  amounts to an invariant spin-tensor wave function, whence the respective field equation is formally the same as the statement (3.91).

For the purpose of deriving the wave equations for gravitons in both formalisms, one may adopt the same basic procedure as that for the electromagnetic situation. In the  $\gamma$ -formalism, we thus allow for the splitting<sup>35</sup>

$$\nabla_{A'}^E \nabla^{AA'} \Psi_{AB}{}^{CD} = \Delta^{AE} \Psi_{AB}{}^{CD} - \frac{1}{2} \gamma^{AE} \square \Psi_{AB}{}^{CD} = 0, \quad (3.92)$$

and account for Eq. (3.34) to perform the calculation

$$\begin{aligned} \Delta^{AE} \Psi_{AB}{}^{CD} &= (X^{AE}{}_M{}^C \Psi_{AB}{}^{MD} + X^{AE}{}_M{}^D \Psi_{AB}{}^{CM}) \\ &\quad - (X^{AE}{}_A{}^M \Psi_{MB}{}^{CD} + X^{AE}{}_B{}^M \Psi_{AM}{}^{CD}) \\ &= 2\chi \Psi^{CDE}{}_B - 3Q^{(CDE)L} \gamma_{LB} \\ &= 2\chi \Psi^{CDE}{}_B - 3Q^{(CDEL)} \gamma_{LB}, \end{aligned} \quad (3.93a)$$

where the auxiliary definition

$$Q^{CDEL} \doteq \Psi_{MN}{}^{CD} \Psi^{ELMN} \quad (3.93b)$$

has been implemented. The legitimacy of the last step of the calculation just taken into consideration stems from the total symmetry of the  $\Psi$ -wave functions, which provides us with the relation

$$Q^{(CDEL)} = Q^{(CDE)L}. \quad (3.93c)$$

Consequently, one is led to the gauge-invariant vacuum equation

$$(\square + 4\chi) \Psi_{AB}{}^{CD} = 6\Psi_{MN}{}^{(CD} \Psi^{EL)MN} \gamma_{EA} \gamma_{LB}. \quad (3.94)$$

The  $\varepsilon$ -formalism version of the splitting (3.92) reads

$$\nabla_{A'}^E \nabla^{AA'} \Psi_{AB}{}^{CD} = \Delta^{AE} \Psi_{AB}{}^{CD} - \frac{1}{2} \varepsilon^{AE} \square \Psi_{AB}{}^{CD} = 0. \quad (3.95)$$

As the index configuration of  $\Psi_{AB}{}^{CD}$  yields a spin-tensor character in both formalisms, we can say that the computation of the  $\Delta$ -derivative of Eqs. (3.95) possesses the same form as that shown above as Eqs. (3.93). It is also clear that any  $\Delta$ -derivatives of  $\Psi_{ABCD}$  within the  $\varepsilon$ -framework carry only gravitational contributions<sup>36</sup> since we are supposedly dealing with a four-index covariant spin-tensor density of weight  $-2$ . It follows that we can write down the  $\varepsilon$ -formalism statement

$$(\square + 4\chi) \Psi_{ABCD} = 6\Psi_{MN(AB} \Psi_{CD)}{}^{MN}. \quad (3.96)$$

<sup>35</sup>The splitting (44) of Ref. [33] carries a miset sign.

<sup>36</sup>This observation is evidently similar to that made previously concerning the  $\varepsilon$ -formalism version of  $\Delta$ -derivatives of  $\phi_{AB}$ .

The  $\gamma$ -formalism pattern for  $\Delta^{AE}\Psi_{ABCD}$  appears as

$$(2\Delta^{AE} + 2i\beta^{EB'}\nabla_{B'}^A - \gamma^{AE}\square)\Psi_{ABCD} = (-4i)\nabla^{EB'}(\beta_{B'}^A\Psi_{ABCD}). \quad (3.97)$$

Some of the pieces of Eq. (3.97) can be manipulated to give the individual contributions

$$2\Delta_E^A\Psi_{ABCD} = 4\chi\Psi_{BCDE} - 6Q_{(BCDE)} + 8i\phi_E^A\Psi_{ABCD}, \quad (3.98a)$$

$$2i\beta_E^{B'}\nabla_{B'}^A\Psi_{ABCD} = 2(\beta^h\beta_h)\Psi_{BCDE}, \quad (3.98b)$$

and

$$(-4i)\nabla_E^{B'}(\beta_{B'}^A\Psi_{ABCD}) = (2\beta^h\beta_h + 4i\beta^h\nabla_h + \Upsilon_{(g)})\Psi_{BCDE} + 8i\phi_E^A\Psi_{ABCD}, \quad (3.98c)$$

with

$$\Upsilon_{(g)} \doteq 2(\beta^h\beta_h + \Upsilon_{(P)}). \quad (3.99)$$

The resulting wave equation is thus written as

$$(\square - 4i\beta^h\nabla_h - \Upsilon_{(g)} + 4\chi)\Psi_{ABCD} = 6\Psi_{MN(AB}\Psi_{CD)}^{MN}. \quad (3.100)$$

Equations (3.94) and (3.100) may be derived from one another by taking account of the prescriptions

$$\square\Psi_{ABCD} = \square(\Psi_{AB}^{LM}\gamma_{LC}\gamma_{MD}), \quad \square(\gamma_{LC}\gamma_{MD}) = -\overline{\Upsilon}_{(g)}\gamma_{LC}\gamma_{MD}, \quad (3.101a)$$

and

$$2(\nabla_a\Psi_{AB}^{LM})\nabla^a(\gamma_{LC}\gamma_{MD}) = 4(2\beta^h\beta_h + i\beta^h\nabla_h)\Psi_{ABCD}. \quad (3.101b)$$

By following up this procedure, we can deduce Eq. (3.100) without having to perform the somewhat lengthy calculations that yield the contributions (3.98). It becomes obvious that the  $\gamma$ -formalism wave equation for  $\Psi^{ABCD}$  may be derived by making use of a similar procedure which takes up the configurations

$$\square\Psi^{ABCD} = \square(\gamma^{AL}\gamma^{BM}\Psi_{LM}^{CD}), \quad \square(\gamma^{AL}\gamma^{BM}) = -\Upsilon_{(g)}\gamma^{AL}\gamma^{BM}, \quad (3.102a)$$

and

$$2\nabla^a(\gamma^{AL}\gamma^{BM})(\nabla_a\Psi_{LM}^{CD}) = 4(2\beta^h\beta_h - i\beta^h\nabla_h)\Psi^{ABCD}. \quad (3.102b)$$

In effect, we have

$$(\square + 4i\beta^h\nabla_h - \overline{\Upsilon}_{(g)} + 4\chi)\Psi^{ABCD} = 6\Psi_{MN}^{(AB}\Psi^{CD)MN}. \quad (3.103)$$

Therefore, as had been established by Cardoso [23], the  $\gamma$ -formalism wave equations satisfied by any fields of valences  $\{a, 0; 0, 0\}$  and  $\{0, a; 0, 0\}$ , as well as their complex-conjugate versions, can be obtained from each other by invoking the interchange rule<sup>37</sup>

$$i\beta^h\nabla_h \leftrightarrow (-i)\beta^h\nabla_h, \quad (\Upsilon_{(P)}, \Upsilon_{(g)}) \leftrightarrow (\overline{\Upsilon}_{(P)}, \overline{\Upsilon}_{(g)}). \quad (3.104)$$

---

<sup>37</sup>This rule supplies the equation  $(\square + 2i\beta^h\nabla_h - \overline{\Upsilon}_{(P)} + \frac{8}{3}\chi)\phi^{AB} = 2\Psi^{AB}{}_{CD}\phi^{CD}$  straightaway from the statement (3.72b). The sourceless wave equations for the  $\varepsilon$ -formalism version of  $\phi_{AB}$  and  $\Psi_{ABCD}$  were derived for the first time in Ref. [15] without taking account of any spin-density characters.

In the presence of sources, one has to rewrite Eq. (3.87) as

$$\nabla_{B'}^A \Psi_{ABCD} - 2i\beta_{B'}^A \Psi_{ABCD} = \frac{\kappa}{2} \nabla_{(B}^{A'} T_{CD)A'B'}. \quad (3.105)$$

It should be noticed that the right-hand side of Eq. (3.105) does not involve the  $T$ -trace piece of Einstein's equations. The sourceful field equation for the  $\gamma$ -formalism  $\Psi$ -wave function of valence  $\{4, 0; 0, 0\}$ , is thus set as

$$\nabla_{AB'} \Psi^{ABCD} + 2i\beta_{AB'} \Psi^{ABCD} = -\frac{\kappa}{2} (\nabla^{A'(B} T^{CD)}_{A'B'} + 2i\beta^{A'(B} T^{CD)}_{A'B'}), \quad (3.106)$$

and, consequently, we also have

$$\nabla_{B'}^A \Psi_{AB}{}^{CD} = \frac{\kappa}{2} \gamma^{CM} \gamma^{DN} \nabla_{(B}^{A'} T_{MN)A'B'}. \quad (3.107)$$

Hence, the  $\varepsilon$ -formalism counterparts of Eqs. (3.105)-(3.107) possess the form

$$\nabla_{B'}^A \Psi_{ABCD} = \frac{\kappa}{2} \nabla_{(B}^{A'} T_{CD)A'B'}, \quad \nabla_{AB'} \Psi^{ABCD} = -\frac{\kappa}{2} \nabla^{A'(B} T^{CD)}_{A'B'}, \quad (3.108)$$

and

$$\nabla_{B'}^A \Psi_{AB}{}^{CD} = \frac{\kappa}{6} (\nabla_B^{A'} T^{(CD)}_{A'B'} + 2\nabla^{A'(C} T_B{}^{D)}_{A'B'}). \quad (3.109)$$

According to the work of Ref. [33], the source contributions to the propagation of gravitons in  $\mathfrak{M}$  can be implemented in the  $\gamma$ -formalism by modifying the wave equations (3.100) and (3.103) to

$$(\square - 4i\beta^h \nabla_h - \Upsilon_{(g)} + \frac{R}{2}) \Psi_{ABCD} - 6\Psi_{MN(AB} \Psi_{CD)}{}^{MN} = -\kappa s_{ABCD}, \quad (3.110a)$$

and

$$(\square + 4i\beta^h \nabla_h - \overline{\Upsilon}_{(g)} + \frac{R}{2}) \Psi^{ABCD} - 6\Psi_{MN}{}^{(AB} \Psi^{CD)MN} = -\kappa s^{ABCD}, \quad (3.110b)$$

where<sup>38</sup>

$$s_{ABCD} = \gamma_{L(A} \nabla_B^{A'} \nabla^{B'L} T_{CD)A'B'} = s_{(ABCD)}. \quad (3.111)$$

In the  $\varepsilon$ -formalism, we correspondingly obtain

$$(\square + \frac{R}{2}) \Psi_{ABCD} - 6\Psi_{MN(AB} \Psi_{CD)}{}^{MN} = -\kappa \nabla_{(A}^{A'} \nabla_B^{B'} T_{CD)A'B'}, \quad (3.112)$$

together with

$$(\square + \frac{R}{2}) \Psi^{ABCD} - 6\Psi_{MN}{}^{(AB} \Psi^{CD)MN} = -\kappa \nabla^{(A|A'} \nabla^{B'|B} T^{CD)}_{A'B'}. \quad (3.113)$$

### 3.4 Wave Equations for Dirac Fields

As in the presentation of world-spin curvature objects, the treatment of Dirac fields as given by Infeld and van der Waerden [3] entirely left out the decompositions that occur in operator-bivector expansions for covariant-differential commutators. The achievement of the spinor computational techniques utilized

<sup>38</sup>It is worth observing that the property  $T_{ab} = T_{(ab)}$  yields  $T_{(CD)A'B'} = T_{(CD)(A'B')}$ .

in the foregoing Subsections is what has really afforded a natural description of the fundamental couplings which should be carried by the wave equations for Dirac fields in  $\mathfrak{M}$  [22]. A notable feature of these configurations is that they are strictly exhibited by the  $\gamma$ -formalism patterns of the statements which control the propagation of the fields. The absence of  $\varepsilon$ -formalism interaction pieces is just due to the spin-vector-density character of the corresponding Dirac fields. Only  $\gamma$ -formalism couplings of electrons and positrons with background photons are brought about by the relevant procedures, there actually occurring no couplings that involve explicit wave functions for gravitons.

The issue concerning the derivation of the couplings between Dirac fields and geometric photons is now entertained. Of course, the spin-curvature splittings of  $\mathfrak{M}$  will be assumed to carry nowhere-vanishing electromagnetic contributions. Like the case of the Infeld-van der Waerden formulation, we think of any Dirac field as a classical wave function whence no specific energy character will be ascribed to it here. The  $\Delta$ -operator prescriptions of Subsection 3.1 will be used so many times that we shall no longer refer to them explicitly.

A Dirac system can be defined in either formalism as the conjugate field pairs borne by the set

$$\mathbf{D} = \{\{\psi^A, \chi_{A'}\}, \{\chi_A, \psi^{A'}\}\}. \quad (3.114)$$

All fields of this set are taken to possess the same rest mass  $m$ . The entries of each pair have the opposite helicity values  $+1/2$  and  $-1/2$ , but such values get reversed when we pass from one pair to the other. In addition, each of the pairs carries the same electric charge, with the charge of one pair being opposite to the charge of the other pair. In the  $\gamma$ -formalism, any element of the set (3.114) behaves as a spin vector under the action of the gauge group. The unprimed and primed elements of the former pair appear in the  $\varepsilon$ -formalism as spin-vector densities of weight  $+1/2$  and antiweight  $-1/2$ , respectively. It is clear that the weights of the  $\varepsilon$ -formalism version of the conjugate fields turn out to be the other way about.

In both formalisms, the theory of Dirac fields was originally taken [3] as the combination of the statements

$$\nabla_{AA'}\psi^A = (-i\mu)\chi_{A'}, \quad \nabla^{AA'}\chi_{A'} = (-i\mu)\psi^A \quad (3.115)$$

with their complex conjugates.<sup>39</sup> In the  $\gamma$ -formalism, the field equations (3.115) are equivalent to

$$\nabla^{AA'}\psi_A = i(\mu\chi^{A'} + \beta^{AA'}\psi_A), \quad (3.116a)$$

and

$$\nabla_{AA'}\chi^{A'} = i(\mu\psi_A + \beta_{AA'}\chi^{A'}). \quad (3.116b)$$

The  $\varepsilon$ -formalism version of Eqs. (3.116) is given by

$$\nabla^{AA'}\psi_A = i\mu\chi^{A'}, \quad \nabla_{AA'}\chi^{A'} = i\mu\psi_A, \quad (3.117)$$

which evidently can be recast into the form of Eqs. (3.115), with the wave functions of the pair  $\{\psi_A, \chi^{A'}\}$  showing up as spin-vector densities of weight  $-1/2$  and antiweight  $+1/2$ . Hence, if we operate with  $\nabla_B^{A'}$  on the first of Eqs.

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<sup>39</sup>The coupling constant borne by Eqs. (3.115) carries the normalized rest mass  $\mu = m/\sqrt{2}$ .

(3.115), thereby implementing the field equation for  $\chi_{A'}$ , we will arrive at the  $\gamma$ -formalism relation

$$(\gamma_{AB}\square - 2\Delta_{AB})\psi^A = (-2)\mu^2\psi_B, \quad (3.118)$$

which amounts to the wave equation

$$(\square + \frac{R}{4} + m^2)\psi^A = (-2i)\phi^A{}_{B'}\psi^{B'}. \quad (3.119)$$

A similar procedure yields the statement

$$(\square + \frac{R}{4} + m^2)\chi_{A'} = 2i\phi_{A'}{}^{B'}\chi_{B'}, \quad (3.120)$$

which accordingly comes out of the configuration

$$(2\Delta^{A'B'} - \gamma^{A'B'}\square)\chi_{A'} = (-2)\mu^2\chi^{B'}. \quad (3.121)$$

The  $\varepsilon$ -formalism counterparts of Eqs. (3.118) and (3.121) involve the purely gravitational derivatives

$$\Delta_{AB}\psi^A = -\frac{R}{8}\psi_B, \quad \Delta^{A'B'}\chi_{A'} = \frac{R}{8}\chi^{B'}, \quad (3.122)$$

whence the corresponding statements are written as<sup>40</sup>

$$(\square + \frac{R}{4} + m^2)\psi^A = 0, \quad (\square + \frac{R}{4} + m^2)\chi_{A'} = 0. \quad (3.123)$$

It becomes manifest that the reason for the non-occurrence of geometric Maxwell-Dirac interactions within the  $\varepsilon$ -framework is related to the weight-valence attributes of the respective Dirac wave functions.

A particular procedure for deriving the  $\gamma$ -formalism wave equations for  $\psi_A$  and  $\chi^{A'}$  consists in allowing suitably indexed  $\nabla$ -operators to act through Eqs. (3.116), likewise taking up either the contravariant differential configuration of Eqs. (3.65c) or its complex conjugate. For  $\psi_A$ , for instance, we thus have the differential relation

$$\Delta^{AB}\psi_A - \frac{1}{2}\gamma^{AB}\square\psi_A = i\nabla_{A'}^B(\mu\chi^{A'} + \beta^{AA'}\psi_A). \quad (3.124)$$

Some calculations similar to those for geometric photons performed anteriorly, supply the following contributions to the right-hand side of Eq. (3.124):

$$i\beta^{AA'}\nabla_{A'}^B\psi_A = \frac{1}{2}(\beta^h{}_{\beta h})\psi^B - i\gamma^{AB}(\beta^h{}_{\nabla h})\psi_A - \mu\beta^{BA'}\chi_{A'}, \quad (3.125)$$

and<sup>41</sup>

$$(\nabla_{A'}^B\beta^{AA'})\psi_A = \frac{1}{2}(\nabla_h\beta^h)\psi^B + 2\phi^{AB}\psi_A. \quad (3.126)$$

Then, implementing the expression

$$\Delta^{AB}\psi_A = \frac{R}{8}\psi^B + i\phi^{AB}\psi_A, \quad (3.127)$$

<sup>40</sup>In the  $\varepsilon$ -formalism, we also have  $(\square + \frac{R}{4} + m^2)\psi_A = 0$  and  $(\square + \frac{R}{4} + m^2)\chi^{A'} = 0$ .

<sup>41</sup>It should be noticed that the computation which yields the right-hand side of Eq. (3.126) actually absorbs one of the relations (3.55).

along with Eq. (3.116b), yields

$$(\square - 2i\beta^h \nabla_h - \Upsilon_{(\mathcal{P})} + \frac{R}{4} + m^2)\psi_A = 2i\phi_A{}^B \psi_B, \quad (3.128)$$

with  $\Upsilon_{(\mathcal{P})}$  being given by the definition (3.72c). For  $\chi^{A'}$ , we likewise obtain the formulae

$$i\beta_{AA'} \nabla_{B'}^A \chi^{A'} = \frac{1}{2}(\beta^h \beta_h) \chi_{B'} + i\gamma_{A'B'}(\beta^h \nabla_h \chi^{A'}) - \mu \beta_{AB'} \psi^A, \quad (3.129)$$

$$(\nabla_{B'}^A \beta_{AA'}) \chi^{A'} = \frac{1}{2}(\nabla_h \beta^h) \chi_{B'} - 2\phi_{A'B'} \chi^{A'}, \quad (3.130)$$

and

$$\Delta_{A'B'} \chi^{A'} = i\phi_{A'B'} \chi^{A'} - \frac{R}{8} \chi_{B'}, \quad (3.131)$$

which give rise to the equation

$$(\square - 2i\beta^h \nabla_h - \Upsilon_{(\mathcal{P})} + \frac{R}{4} + m^2) \chi^{A'} = (-2i)\phi^{A'}{}_{B'} \chi^{B'}. \quad (3.132)$$

The consistency between the  $\gamma$ -formalism wave equations we have exhibited may be verified by taking into account the prescriptions

$$\square \gamma^{BC} = -\Upsilon_{(\mathcal{P})} \gamma^{BC}, \quad \square \gamma_{BC} = -\overline{\Upsilon}_{(\mathcal{P})} \gamma_{BC}, \quad (3.133a)$$

and

$$\square \psi^A = \gamma^{AB} \square \psi_B + (\square \gamma^{AB}) \psi_B + 2(\nabla^h \gamma^{AB}) \nabla_h \psi_B, \quad (3.133b)$$

along with

$$\Delta_{AB} \psi_C - \gamma_{MC} \Delta_{AB} \psi^M = 2i\phi_{AB} \psi_C, \quad (3.134a)$$

and

$$\Delta_{A'B'} \chi_{C'} - \gamma_{M'C'} \Delta_{A'B'} \chi^{M'} = (-2i)\phi_{A'B'} \chi_{C'}. \quad (3.134b)$$

It can then be said that the right-hand sides of such wave equations are the only structures which carry the interaction patterns produced by the propagation in  $\mathfrak{M}$  of the fields borne by the pairs  $\{\psi^A, \chi_{A'}\}$  and  $\{\psi_A, \chi^{A'}\}$ . We point out that these patterns are not affected by the implementation of any devices for changing valence configurations like the ones of Eqs. (3.133) and (3.134).

## 4 Conclusions and Outlook

The only spacetime-metric character of the  $\varepsilon$ -formalism is exhibited by Eqs. (2.7b) and (2.68), which thus yield the expressions

$$\mathfrak{e} = K(-\mathfrak{g})^{-1/2}, \quad \Sigma_h^{BB'} \partial_a \Sigma_{BB'}^h = \partial_a \log \mathfrak{e},$$

where  $K$  stands for a positive-definite world-spin invariant. An  $\varepsilon$ -formalism counterpart of Eq. (2.90) can therefore be brought into the overall metric picture, according to the requirement

$$\nabla_a \mathfrak{e} = 0.$$

In addition, the transformation law (2.104) suggests the implementation of a prescription of the type

$$\Pi_a = \partial_a \log(|E|^{-1}) \Rightarrow \partial_{[a} \Pi_{b]} = 0,$$

with  $|E|$  amounting to a covariantly constant world-invariant spin-scalar density of absolute weight +1 that carries no specific metric meaning. This prescription can be considered as a formal counterpart of Eq. (2.43), which is merely associated to the spin-displacement configuration

$$\Pi_a dx^a = d \log(|E|^{-1}).$$

Its utilization guarantees the genuineness of the  $\varepsilon$ -formalism version of Eq. (3.1) through

$$\nabla_{[a} (\Pi_{b]} \Sigma^{cDD'}) = \Sigma^{cDD'} \partial_{[a} \Pi_{b]} = 0.$$

In the  $\gamma$ -formalism, the presence or absence of intrinsically geometric electromagnetic fields is traditionally controlled by means of the metric devices provided by Eqs. (2.73). In fact, the derivatives (3.48) and (3.49) supply alternative “electromagnetic switches” of the form

$$\Delta_{AB} \gamma_{CD} = (\Delta_{AB} \gamma) \varepsilon_{CD} = (2i\phi_{AB}) \gamma_{CD}.$$

Then, whenever  $\Phi_a$  is taken as a gradient, we may allow for the relationship

$$(-2)\phi_{AB} = \Delta_{AB} \Phi = 0,$$

which obviously brings out the torsionlessness of  $\nabla_a$  as expressed by

$$[\nabla_a, \nabla_b] \Phi = 0.$$

Another noteworthy difference between the formalisms is related to the non-availability of any  $\varepsilon$ -counterparts of such electromagnetic devices.

A gauge-covariant form of the limiting procedure gets clearly exhibited when we call for the  $\gamma\varepsilon$ -formulae

$$\Theta_{aBC}^{(\gamma)} = \gamma \Theta_{aBC}^{(\varepsilon)},$$

and

$$\Gamma_{A(BC)A'(B'C')}^{(\gamma)} = | \gamma |^3 \Gamma_{A(BC)A'(B'C')}^{(\varepsilon)},$$

together with

$$\sigma_{h(B}^{D'} \partial_{|a|} \sigma_{C)D'}^h = \gamma \Sigma_{h(B}^{D'} \partial_{|a|} \Sigma_{C)D'}^h,$$

and

$$\gamma_{a(BC)} = \gamma \Gamma_{a(BC)}.$$

Such structures can be used to show that the definition (3.3) for the  $\gamma$ -formalism equals its  $\varepsilon$ -formalism counterpart, that is to say,

$$W_{abA}^{(\gamma)B} = W_{abA}^{(\varepsilon)B} \Leftrightarrow W_{abAB}^{(\gamma)} = \gamma W_{abAB}^{(\varepsilon)}.$$

Consequently, one can write the configuration

$$\vartheta_{aBC} = \frac{1}{2} (S_{(B}^{D'} \partial_{C)D'} g_{ab} + S_{b(B}^{D'} \partial_{|a|} S_{C)D'}^b + \vartheta_{aD}^D M_{BC}),$$

together with Eq. (2.61) and the explicit  $\gamma$ -formalism expression

$$\gamma_{aB}{}^B = \frac{1}{4}(\Gamma_a + \sigma_h^{BB'} \partial_a \sigma_{BB'}^h) - 2i\Phi_a.$$

We thus have been able to build up a metric expression for  $\gamma_{aBC}$  and likewise to construct out of employing the limiting procedure the corresponding configuration for  $\Gamma_{aBC}$ . The combination of these results with the relations

$$R_{AA'BB'CC'DD'}^{(\gamma)} = |\gamma|^4 R_{AA'BB'CC'DD'}^{(\varepsilon)},$$

$$\omega_{ABCD}^{(\gamma)} = \gamma^2 \omega_{ABCD}^{(\varepsilon)},$$

and

$$\omega_{A'B'CD}^{(\gamma)} = |\gamma|^2 \omega_{A'B'CD}^{(\varepsilon)},$$

establishes once and for all the invariant equality  $\chi^{(\gamma)} = \chi^{(\varepsilon)}$ , and additionally enhances the correspondence principle involved in the limiting process.

A transparent way of characterizing  $\Phi_a$  and  $\varphi_a$  as affine electromagnetic potentials is afforded by the commutators that yield the curvature spinors of  $\gamma_{aB}{}^C$  and  $\Gamma_{aB}{}^C$ . The  $W$ -objects for both formalisms also arise from the combination of Eq. (3.1) with either of the commutators

$$[\nabla_a, \nabla_b]u^{CC'} = S_c^{CC'} R_{abh}{}^c u^h,$$

and

$$[\nabla_a, \nabla_b]u_{CC'} = -S_{CC'}^c R_{abc}{}^h u_h.$$

Suitably contracted versions of these structures lead to purely gravitational configurations like

$$\Delta_{AB}u^{BC'} = \Xi_{ABD'}{}^{C'} u^{BD'} - \frac{R}{8} u_A{}^{C'},$$

whence, in either formalism, we may implement the Hermitian expansions

$$[\nabla_a, \nabla_b]u^{CC'} = W_{abD}{}^C u^{DC'} + \text{c.c.},$$

and

$$[\nabla_a, \nabla_b]u_{CC'} = -(W_{abC}{}^D u_{DC'} + \text{c.c.}).$$

We can attain a confirmation of the result regarding the tensor behaviour of the  $\gamma$ -formalism wave equations for gravitons and geometric photons by invoking the gauge invariance of  $\beta_a$  along with the transformation law for  $\Theta$  and the homogeneous pattern

$$\square'(\Omega' T'_{B\dots C}) = (\Delta_\Lambda)^a (\bar{\Delta}_\Lambda)^b | \Delta_\Lambda |^c \Lambda_B^L \dots \Lambda_C^N \square(\Omega T_{L\dots N}).$$

This procedure takes up implicitly the gauge invariance of the  $\Upsilon$ -functions defined as Eqs. (3.72c) and (3.99). Evidently, the entire  $\gamma$ -formalism system of electromagnetic wave equations could have been made up by allowing for a procedure which looks like the gravitational one exhibited by Eqs. (3.101) and (3.102).

The implementation of specific techniques for solving Eq. (3.70a) would become considerably simplified if the situations being entertained were set upon

conformally flat spacetimes. Under such a circumstance, one would just deal in either formalism with the equation

$$(\square + \frac{R}{3})\phi_A{}^B = 0,$$

which can be treated by utilizing the invariant distributional methods given by Friedlander [34]. We believe that, within the standard Friedmann-Robertson-Walker cosmological framework [35, 36], solutions to this equation satisfying suitably prescribed boundary conditions should describe some of the properties of the cosmic microwave background. In particular, the energy-momentum spin tensor

$$T_{AA'}{}^{BB'} = \frac{1}{2\pi}\phi_A{}^B\phi_{A'}{}^{B'},$$

would presumably be helpful for achieving present-time values of the energy of the radiation.

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## References

- [1] Van der Waerden, B.L.: Nachr. Akad. Wiss. Göttingen, Math. Physik. Kl., 100 (1929)
- [2] Infeld, L.: Physik ZS. **33**, 475 (1932)
- [3] Infeld, L., Van der Waerden, B.L.: Sitzber. Akad. Wiss., Physik-math. Kl. **9**, 380 (1933)
- [4] Weyl, H.: Z. Physik **56**, 330 (1929)
- [5] Schouten, J.A.: Jour. of Math. and Phys. **10**, 239 (1931)
- [6] Schouten, J.A.: Z. Physik **84**, 92 (1933)
- [7] Cardoso, J.G.: Nuovo Cimento B **5**, 575 (1996); New Two-Component Spinor Formulae for Classical General Relativity. In: Piran, T. (ed.) WS Proceedings of the 8th Marcel Grossmann Meeting, Jerusalem, p. 641. WS (1997); Int. Jour. Theor. Phys. (in press)
- [8] Penrose, R., Rindler, W.: Spinors and Space-Time Vol. 1, Cambridge (1984)
- [9] Penrose, R., Rindler, W.: Spinors and Space-Time Vol. 2, Cambridge (1986)
- [10] Laporte, O., Uhlenbeck, G.E.: Phys. Rev. **37**, 1380 (1949)
- [11] Bade, W.L., Jehle, H.: Rev. Mod. Phys. **3**, 714 (1953)
- [12] Corson, E.M.: Introduction to Tensors, Spinors and Relativistic Wave Equations, Glasgow (1953)

- [13] Bergmann, P.G.: Phys. Rev. **2**, 624 (1957)
- [14] Witten, L.: Phys. Rev. **1**, 357 (1959)
- [15] Penrose, R.: Ann. Phys. (N.Y.) **10**, 171 (1960)
- [16] Petrov, A.Z.: Einstein Spaces, Oxford (1969)
- [17] Newman, E.T., Penrose, R.: Jour. Math. Phys. **3**, 566 (1962)
- [18] Geroch, R., *et all.*: Jour. Math. Phys. **14**, 874 (1973)
- [19] Ludwig, G.: Int. Jour. Theor. Phys. **27**, 3, 315 (1988)
- [20] Kolassis, C., Chan, R.: Class. Quant. Grav. **6**, 697 (1989)
- [21] Penrose, R.: Acta Phys. Polon. **10**, 2979 (1999)
- [22] Cardoso, J.G.: Class. Quant. Grav. **23**, 4151 (2006)
- [23] Cardoso, J.G.: Czech Journal of Physics **4**, 401 (2005)
- [24] Cardoso, J.G.: Acta Phys. Polon. B **8**, 1001 (2007)
- [25] Bach, R.: Math. Zeitschr. **9**, 110 (1921)
- [26] Schouten, J.A.: Indagationes Math. **11**, 178; 217; 336 (1949)
- [27] Schouten, J.A.: Ricci Calculus, Heidelberg (1954)
- [28] Cardoso, J.G.: Jour. Math. Phys. **1**, 0425041 (2005)
- [29] Pauli, W.: Relativity Theory, London (1958)
- [30] Schrödinger, E.: Space-Time Structure, Cambridge (1963)
- [31] Landau, L.D., Lifchitz, L.: Théorie du Champ, Moscou (1966)
- [32] Carmeli, M., Malin, S.: Theory of spinors, An Introduction, Singapore (2000)
- [33] Cardoso, J.G.: Il Nuovo Cimento B **6**, 631 (2009)
- [34] Friedlander, F.G.: The Wave Equation in a Curved Spacetime, Cambridge (1975)
- [35] Padmanabhan, T.: Physics Reports **380**, 235 (2003)
- [36] Bartelmann, M.: Rev. Mod. Phys. **82**, 331 (2010)